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PhD Thesis

Singularities in thin magnetic films

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Chapter 1

Introduction

In this thesis we will study some mathematical properties related to the micromagnetics of thin films. Micromagnetics is the study of ferromagnetic materials at length scales large enough so that we can ignore the atomic structure of the material (and so the quantum effects) but small enough (typically at the level of sub-micrometers) that interesting patterns and microstructures arise. The behaviour of a ferromagnetic body $\boldsymbol{\omega} \subset \mathbb{R}^3$ is described by a 3-dimensional unit vector field $\mathbf{m} : \boldsymbol{\omega} \to \mathbb{S}^2$, called the *magnetization*, whose stable states correspond to the local minima of the following energy functional, called the *micromagnetic energy*:

$$E(\mathbf{m}) = d^2 \int_{\boldsymbol{\omega}} |\nabla \mathbf{m}|^2 d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla U|^2 d\mathbf{x} + Q \int_{\boldsymbol{\omega}} \varphi(\mathbf{m})\, d\mathbf{x} - \int_{\boldsymbol{\omega}} \mathbf{m} \cdot \mathbf{h}_{ext} d\mathbf{x}. \quad (1.1)$$

1. The first term is the *exchange energy*, which penalizes spatial variations of the magnetization on the length scale of the *exchange constant d*. This is an intrinsic parameter of the material, of the order of nanometres. This energy term models the tendency of electron spins to align in a constant direction.

2. The second term is the *stray-field energy* (also called the *magnetostatic*

energy) and it is the energy of the magnetic field generated by the magnetization $\mathbf{m}$ in the whole space. The *stray-field potential* $U : \mathbb{R}^3 \to \mathbb{R}$ is a solution to the static Maxwell equation:

$$\nabla \cdot (\nabla U + \mathbf{m}\mathbb{1}_\omega) = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{1.2}$$

We can express this in a different way saying that the stray field $-\nabla U$ is the L^2 Helmholtz projection onto the gradient fields of the magnetization (extended by 0 outside of $\boldsymbol{\omega}$). The magnetostatic equation favours divergence-free magnetizations: indeed, if $\mathbf{m}$ minimizes the magnetostatic energy this implies that the corresponding stray-field potential is harmonic. Now equation (1.2) implies that $\mathbf{m}$ is divergence-free. We furthermore notice that this term is non-local, which adds further complexity to the problem, compared to the case of the Ginzburg-Landau energy, which is local.

3. The *anisotropy energy* $Q \int_\omega \varphi(\mathbf{m})\, d\mathbf{x}$ models the tendency of the magnetization to align along special directions which depend on the crystalline structure of the material. The latter is described by the function φ (which is usually polynomial and whose zeroes represent the favoured directions for the magnetization, called, depending on the dimension of the zero set, *easy axes* or *easy planes*). Its strength relative to the the other energy terms is controlled by the *quality factor* Q. We distinguish between *soft* ($Q < 1$) and *hard* ($Q > 1$) materials.

4. The last term is the *Zeeman energy*, which comes from the presence of an external magnetic field $\mathbf{h}_{ext}$. It models the tendency of the magnetization to align with such field.

The energy functional of micromagnetics was first introduced by Landau and Lifshitz [39] in 1935, even though the roots of micromagnetics as a proper field of enquiry are to trace back to William Fuller Brown with the publication of his book ‘Micromagnetics’ in 1963 [10]. For a good introduction to micromagnetics from a physical point of view we refer for instance to Arrott [4], which is very thorough and displays a wealth of figures and graphs. Although the study of micromagnetics from the point of view of physics dates back to more than 80 years ago, its mathematical treatment is much more recent.

The first analytical approaches to this subject where principally based on finding solutions by minimizing the energy over a suitable class of ansatz fields: as noted in [31], the classic example for this approach is the Stoner-Wohlfarth theory of magnetic switching [50]. Another line of studies has approached the subject from a computational point of view, which has been very successful in its own way (we can recall for example the simulations of [24]). Both these approaches have their advantages, but also their limitations: for example the approach via an ansatz is limited by the correctness of such ansatz, and some of the solutions obtained in this way have turned out later to be wrong (albeit still informative on the behaviour of minimizers), as pointed out in [31]. The already mentioned Stoner-Wohlfarth theory matches the experiments only in a very limited range, for very small particles. Numerical simulation can help overcome some of the difficulties that the ansatz based approach presents, and has in fact turned out to be a valuable instrument to answer a variety of specific questions on patterns and microstructures. However, this also is limited in that it does not provide a way to understand these phenomena: for instance it is not helpful if we want to explain why such a variety of microstructure arises in different regimes, for example why in some regimes we only have boundary vortices, and no inner ones, or why in some other regimes we observe the formation of domain walls. Therefore an analytical approach is needed to gain a deeper understanding of some of the questions that the other approaches leave unanswered, and to complement their findings. Sometimes the numerical simulation can give some hints that can be confirmed through analytical means: this will be the content of Chapter 6 where we prove rigorously the result that for a specific regime of thin-film micromagnetics the so-called S state has lower energy than the so-called C state, thus confirming the numerical experiments of Kohn and Slastikov [32]. An interesting field of study in micromagnetics comes from the analysis of so-called micromagnetic *microstructures*: these include vortices (both in the interior and on the boundary), domain walls (see for example Kurzke et al. [38], Wang et al. [56], Lund and Muratov [42] or Lund et al. [43]) and skyrmions (see for example Troncoso and Núñez [54] and Walton [55]), to give some examples.

In this thesis we will focus on thin-films, that is we will consider domains of the form $\boldsymbol{\omega} := \omega \times (0, t)$, where $\omega \subset \mathbb{R}^2$ has diameter $\operatorname{diam}(\omega) = l$ and t is very small (we will make this more precise below by specifying which asymptotic regime we will consider, see (1.10)). We see that, in its full version, our problem is non-local (due to the stray-field energy), non-convex (due to the unit length constraint on the magnetization) and it depends on four parameters, two *intrinsic*, which are given by the material (the exchange length and the quality factor), and two *extrinsic*, which are determined by the shape of the sample (the thickness and the diameter of the cross-section). In our study we will neglect the effect due to anisotropy (i.e. we will assume $Q = 0$) and to the presence of an external field (i.e. we will assume $\mathbf{h}_{ext} = 0$).

Since we study thin films, a natural attempt is to try to reduce the full three-dimensional model to a two-dimensional one, considering the limit in which the thickness tends to 0 (for a brief introduction to some dimension-reduction problems, see for example the excellent book by Braides [9, Chapter 14], which also provides a very good introduction to Γ-convergence). We are neglecting the effects due to anisotropy, therefore our problem is characterized by three parameters, all of which have the dimension of a length: the in-plane diameter l, the thickness t and the exchange length d. We rescale with respect to the diameter of the cross section to get the following two dimensionless parameters:

$$h := \frac{t}{l} \quad \text{(aspect ratio)}, \quad \eta := \frac{d}{l} \quad \text{(normalized exchange length)}. \tag{1.3}$$

We also introduce the following new parameter ε, which represents the core size of a boundary vortex[1]:

$$\varepsilon = \frac{\eta^2}{h|\log h|}. \tag{1.4}$$

We then nondimensionalize with respect to the base diameter l, i.e. we consider the new quantity $\hat{\mathbf{x}} := \frac{\mathbf{x}}{l}$, where

$$\hat{\mathbf{x}} \in \boldsymbol{\Omega}_h := \Omega \times (0, h) \subset \mathbb{R}^3, \tag{1.5}$$

where $\Omega = \frac{\omega}{l} \subset \mathbb{R}^2$ (by definition, this has diameter one), along with $\mathbf{m}_h(\hat{\mathbf{x}}) = \mathbf{m}(\mathbf{x})$, $U_h(\hat{\mathbf{x}}) = \frac{1}{l} U(\mathbf{x})$. We then renormalize the energy as follows:

$$\hat{E}_h(\mathbf{m}_h) = \frac{1}{d^2 t |\log \varepsilon|} E(\mathbf{m}). \tag{1.6}$$

From now on, since we will only work with this energy, we will drop the hat for simplicity and we will therefore consider the energy:

$$E_h(\mathbf{m}_h) = \frac{1}{h|\log \varepsilon|} \int_{\boldsymbol{\Omega}_h} |\boldsymbol{\nabla} \mathbf{m}_h|^2 d\mathbf{x} + \frac{1}{\eta^2 h |\log \varepsilon|} \int_{\mathbb{R}^3} |\boldsymbol{\nabla} U_h|^2 d\mathbf{x}, \tag{1.7}$$

where the new quantities that we have introduced still satisfy all the relevant properties, i.e.:

$$\mathbf{m}_h : \boldsymbol{\Omega}_h \to \mathbb{S}^2, \quad \boldsymbol{\Delta} U_h = \boldsymbol{\nabla} \cdot \left(\mathbf{m}_h \mathbb{1}_{\boldsymbol{\Omega}_h}\right) \text{ in } \mathbb{R}^3. \tag{1.8}$$

It is the multi-scale nature of the problem, combined with its non-convex and non-local character, which makes it so rich and interesting, since it exhibits a

[1] A boundary vortex is a small region of the boundary where the magnetization undergoes a change from being almost parallel to being almost antiparallel to the tangent vector field.

variety of patterns, according to the mutual relationships between these parameters. Because we are studying thin films, it is natural to consider regimes in which $h \ll 1$.

- The first rigorous mathematical study of thin-films was conducted by Gioia and James [21] in the regime in which $h \to 0$ and η stays fixed (see also Kreisbeck [33]). In this case the Γ-limit is somewhat degenerate, because it is minimized by all constant and in-plane magnetizations, so it does not give much useful information, and also does not take into account the shape of the sample.

- The regime $h \to 0$ and $\eta^2 \ll \frac{h}{|\log h|}$ has been studied by De Simone, Kohn, Müller and Otto [16]. Their main result is the Γ-convergence of the suitably rescaled three-dimensional functionals to some two-dimensional reduced energy, in the presence of a specified external field. This aims to reproduce the gross features observed in experiments, where the magnetization does not depend on the thickness direction, has no out-of-plane component and is divergence free in the absence of an external field. They seek a reduced theory with a single length scale. This approach is analogous to what we will do in this thesis.

- Kohn and Slastikov [32], drawing on work by Carbou [13], studied the two cases where $\eta^2 \gg h|\log h|$ and $\eta^2 \sim h|\log h|$, which correspond to very small (soft) films and slightly larger ones respectively, with no applied magnetic field.

  In the former case the exchange energy is dominant and the resulting limit magnetization is constant and in-plane. The magnetostatic energy reduces to a local contribution on the boundary, given by $\int_{\partial\Omega} (m \cdot \nu)^2$.

  In the latter case the exchange and the magnetostatic energies compete and the Γ-limit functional is given by

$$E_{KS}^{\alpha} := \alpha \int_{\Omega} |\nabla m|^2 + \frac{1}{2\pi} \int_{\partial\Omega} (m \cdot \nu)^2 \,, \tag{1.9}$$

where $\alpha = \lim_{h\to 0} \frac{\eta^2}{h|\log h|}$, subject to the constraints $m_3 = 0$ and $|m| = 1$. In this case, unlike in the work of Carbou [13] and Gioia and James [21], the optimal m does not have to be constant, but the asymptotic contribution of the magnetostatic energy is still given by a constant times $\int_{\partial\Omega} (m \cdot \nu)^2$.

- The limit case for $\alpha \to 0$ of $\frac{1}{\alpha} E_{KS}^{\alpha}$ was studied by Kurzke [35]: the author shows the formation of boundary vortices and proves convergence of minimizers. The fact that we observe the formation of boundary vortices is due to the fact that there is no $m \in H^1(\Omega, \mathbb{S}^1)$ such that $m \cdot \nu = 0$ on $\partial\Omega$ if Ω is simply connected. The leading order term in the energy expansion is related to the number of vortices. The author also shows that the first non-singular term of the energy is given by a *renormalized energy* which depends only on the position of the vortices and their degrees, and describes the interaction of the vortices. The gradient flow corresponding to this energy was studied by the same author in [37].

- The regime $\eta^2 = O(h)$ was studied by Moser [44, 45]. The author studies convergence of minimizers in suitable spaces and finds an explicit representation for the limit function with boundary vortices, and provides equations satisfied by the limit. In [45] he also studies the dynamics of such vortices, i.e. he studies solutions of the corresponding Landau-Lifshitz equation.

- In a paper soon to appear, Ignat and Kurzke [25] proved rigorously the connection between micromagnetics and the model studied by Kurzke [35] in the regime in which $\eta, h \ll 1$ and $h \ll \eta^2 \ll h|\log h|$ for a $C^{1,1}$ domain. In this thesis we will prove the same results for rectangular domains.

In this thesis we will work in the same regime considered by Ignat and Kurzke [25] of thin-films (i.e. in which $h \ll 1$), where we expect the energy to concentrate around boundary vortices. More precisely we consider the following regime:

$$\eta, h \ll 1 \quad \text{and} \quad \frac{1}{|\log h|} \ll \varepsilon \ll 1, \tag{1.10}$$

where $\eta = \eta(h)$ and $\varepsilon = \varepsilon(h)$ are defined as in (1.3) and (1.4). Multiplying by $h|\log h|$ we see that the second relation in (1.10) is equivalent to $h \ll \eta^2 \ll h|\log h|$, from which we can conclude that $|\log h| \sim |\log \eta|$. We can then rewrite the regime in terms of the new parameters ε and η as:

$$\eta \ll 1 \quad \text{and} \quad \frac{1}{|\log \eta|} \ll \varepsilon \ll 1. \tag{1.11}$$

1.1 Overview of results

The main focus of this thesis is the study of micromagnetic properties of rectangular thin-films. We will study the regime for the parameters defined in (1.10) and show that the full energy (1.1), which is vector-valued, non-local, and defined on a three-dimensional domain (the thin-film) can be replaced by a scalar-valued, local energy defined on a two-dimensional domain (the base of the thin-film), namely the energy studied by Kurzke in his PhD thesis [34]. We consider a family $\mathbf{m}_h$ of magnetizations of bounded energy, i.e. for which $\limsup_{h\to 0} E_h(\mathbf{m}_h) < \infty$. In Chapter 2 we show that we can reduce our three-dimensional model to a local model in two-dimensions, by averaging the magnetization in the out-of-plane direction, in a way that does not increase the energy asymptotically. More precisely, we introduce a new so-called *reduced energy* $\overline{E}_h$ and we show that the reduced energy of the averaged magnetization (which we denote by $\overline{\mathbf{m}}_h$) and the micromagnetic energy of the original family of magnetizations are close asymptotically, or in other words $E_h(\mathbf{m}_h) \geq \overline{E}_h(\overline{\mathbf{m}}_h) - o(1)$ as $h \to 0$. The key element in this proof is to replace the non-local magnetostatic energy with a local one which does not depend on the out-of-plane component of the magnetization and which has the form:

$$\frac{1}{|\log \varepsilon|} \left(\int_{\Omega} \frac{1}{\eta^2} \left(1 - |\overline{m}_h|^2\right)^2 dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1 \right).$$

In the Chapter 3 we then proceed to show that we can replace the averaged magnetization (or more precisely the in-plane component thereof) by a unit length vector M_h, again in a way that does not increase the energy asymptotically (here we do not need to introduce any new energy, we just compare the reduced energies of the averaged magnetizations and the corresponding unit-length vectors). This makes the first term in the reduced magnetostatic energy vanish and leaves us with (modulo a multiplicative factor of $|\log \varepsilon|$):

$$\frac{1}{2\pi\varepsilon} \int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1.$$

By considering now functions u_ε such that $M_h = e^{iu_\varepsilon}$ and g such that the tangent vector τ can be written as $\tau = e^{ig}$ we can rewrite this term as:

$$\frac{1}{2\pi\varepsilon} \int_{\partial\Omega} \sin^2\left(u_\varepsilon - g\right) d\mathcal{H}^1,$$

and we recover the functionals that have been studied before by Kurzke in his PhD thesis [34] and subsequent articles (see for instance [35], [37], which we go on to study in Chapters 5, 6 and 7. We first prove lower and upper bounds for the energy of minimizers in rectangles in Chapter 5, which we use in Chapter 6 to prove convergence of minimizers to harmonic functions with boundary jumps in the corners. In the same chapter we also prove an energy expansion for such limit functions and prove that a certain limit configuration (the so-called S state) has minimal energy: we do this by proving an explicit expression for the energy in the different configurations, which we can then compare. In Chapter 7 we prove second order lower bounds for the energy: to do this the crucial result needed is a uniqueness theorem for solutions to the Euler-Lagrange equation (properly rescaled) in a neighbourhood of a corner. The rescaled equation in a quadrant $Q := \{x, y > 0\}$ is

$$\begin{cases} \Delta u = 0 & \text{in} \quad Q \\ \frac{\partial u}{\partial \nu} = \frac{1}{2} \sin 2u & \text{on} \quad x = 0 \\ \frac{\partial u}{\partial \nu} = -\frac{1}{2} \sin 2u & \text{on} \quad y = 0. \end{cases}$$

We know a particular solution of this, namely the function

$$u(x, y) := \arctan\left(\frac{y+1}{x+1}\right).$$

Our uniqueness result says that given some suitable conditions (more about this in Chapter 4) this solution is unique. These conditions appear naturally in Chapter 7 where we apply this uniqueness result, and so are entirely reasonable to assume. This is the main result of Chapter 4, where we also prove an energy expansion for such a solution. At the end of Chapter 7 we then return to the full model in three-dimension and prove lower bounds for the full micromagnetic energy. For a sequence $\mathbf{m}_h$ with uniformly bounded energy (i.e. such that $\limsup_{h\to 0} E_h(\mathbf{m}_h) \leq C$) we have the lower bound

$$\liminf_{h\to 0} E_h(\mathbf{m}_h) \geq 2\pi.$$

We can also prove a second order lower bound in the more restrictive regime (see Chapter 7 for more about this), in terms of a finite dimensional renormalized energy. We then construct suitable recovery sequences that allow us to formulate a Γ-convergence result for the full energy. More precisely we can construct a sequence $\mathbf{M}_h \in H^1(\mathbf{\Omega}_h; \mathbb{S}^1)$ such that

$$\lim_{h\to 0} E_h(\mathbf{M}_h) = 2\pi.$$

With this we conclude our analysis of micromagnetics in rectangular thin films. In the last chapter – unrelated to the rest of the thesis – we prove a single multiplicity result for boundary vortices for the scalar functionals studied by Kurzke [35]. In his work he proved that the degree of vortices for minimizers can only be ± 1. We prove that, under the assumption of a logarithmic upper bound for the energy, the same result holds true for critical points which are not minimizers.

1.2 Examples

In order to better understand interior and boundary vortices and to see why we expect that in the case that we consider there will only be boundary vortices, we will now give a brief illustration of what these microstructures typically look like and what their energy contribution is.

1.2.1 Interior vortex

The typical example of an interior vortex is given by a vector field

$$\mathbf{m} = (m, m_3) : \Omega \to \mathbb{S}^2 \tag{1.12}$$

which minimizes the reduced energy on the unit disk $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$, subject to the boundary condition $\mathbf{m} = \tau$ on $\partial\Omega$:

$$\overline{E}_h(\mathbf{m}) = \frac{1}{|\log \varepsilon|} \left(\int_\Omega |\nabla \mathbf{m}|^2 dx + \frac{1}{\eta^2} \int_\Omega \left(1 - |m|^2\right) dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (m \cdot \nu)^2 \, d\mathcal{H}^1 \right). \tag{1.13}$$

In this case, because of the boundary condition, the energy becomes

$$\overline{E}_h(\mathbf{m}) = \frac{1}{|\log \varepsilon|} \left(\int_\Omega |\nabla \mathbf{m}|^2 dx + \frac{1}{\eta^2} \int_\Omega \left(1 - |m|^2\right) dx \right). \tag{1.14}$$

The boundary condition means that the magnetization turns in-plane at the boundary and therefore it carries a topological degree $\deg(m, \partial\Omega) = 1$. At the center of the disk we witness the formation of a localized region, the core of the vortex, at which center the magnetization becomes vertical, i.e. $\mathbf{m} = (0, m_3)$, and it carries a polarity ± 1, depending on the sign of m_3. The rescaled reduced energy can be compared to the Ginzburg-Landau energy E_{GL} (defined as $E_{GL}(u) := \int_\Omega \left[|\nabla u|^2 + \frac{1}{\varepsilon^2}\left(1 - |u|^2\right)^2 \right] dx$) as follows:

$$\begin{aligned} |\log \varepsilon| \overline{E}_h(\mathbf{m}) &= \int_\Omega |\nabla \mathbf{m}|^2 dx + \frac{1}{\eta^2} \int_\Omega \left(1 - |m|^2\right) dx \\ &\geq \int_\Omega |\nabla m|^2 dx + \frac{1}{\eta^2} \int_\Omega \left(1 - |m|^2\right)^2 dx = E_{GL}(m), \end{aligned} \tag{1.15}$$

where we used that $|\nabla \mathbf{m}|^2 \geq |\nabla m|^2$ and $1 - |m|^2 \geq \left(1 - |m|^2\right)^2$. For a minimizer we have that this inequality is indeed an equality up to an additive constant which does not depend on $\eta \to 0$ (see Ignat and Otto [28]) and so we can express the minimal energy using Ginzburg-Landau theory as:

$$\min_{m = \tau \mathrm{on} \partial\Omega} \overline{E}_h(m) = \frac{2\pi |\log \eta|}{|\log \varepsilon|} + O\left(\frac{1}{|\log \varepsilon|}\right) \quad \text{as} \quad \varepsilon, \eta \to 0. \tag{1.16}$$

Observe that in our regime we have $|\log \varepsilon| \ll |\log \eta|$, so an interior vortex will asymptotically have infinite energy. For sequences of magnetizations with uniformly bounded energy as those we will consider, this indicates that we do not expect to have interior vortices.

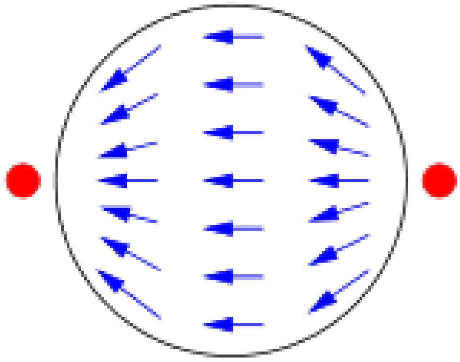
1.2.2 Boundary vortex

An example of a typical boundary vortex is given by an $\mathbb{S}^1$-valued vector field m on the unit disk Ω (i.e. a magnetization $\mathbf{m}$ for which we have $m_3 = 0$) that minimizes the reduced energy (1.13), which in this case, due to the unit length constraint, becomes

$$\overline{E}_h(\mathbf{m}) = \frac{1}{|\log \varepsilon|} \left(\int_\Omega |\nabla \mathbf{m}|^2 dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (m \cdot \nu)^2 \, d\mathcal{H}^1 \right). \tag{1.17}$$

This problem has been studied by Kurzke [35, 36] and Moser [44]: the minimizer is a harmonic function which has two boundary vortices of degree 1/2, diametrically opposed. A boundary vortex is given by region of size ε where the

Figure 1.1: A pair of boundary vortices in the disk.



magnetization transitions in-plane from $-\tau$ to τ. If we consider a lift u of m (i.e. a function u such that $m = e^{iu}$, then this means that u there changes rapidly, with a jump that asymptotically is $-\pi$. The energy cost of such a transition is given by:

$$\frac{1}{2}\overline{E}_h(m) = \pi + O\left(\frac{1}{|\log\varepsilon|}\right), \tag{1.18}$$

so that the total energy is

$$\min_{m \in H^1(\Omega, \mathbb{S}^1)} \overline{E}_h(m) = 2\pi + O\left(\frac{1}{|\log\varepsilon|}.\right). \tag{1.19}$$

Thus a family of magnetizations with uniformly bounded energy can have boundary vortices in our regime. The study of boundary vortices in this case has been conducted by Ignat and Kurzke [25, 26].

1.3 Outlook

In this thesis we have generalized the results of Ignat and Kurzke [25] to the case of rectangles. It remains an interesting open question to study the problem for general convex polygons, or more generally for arbitrary bounded Lipschitz domains. Some of the results which we used for rectangles can be carried over to general polygons, but some crucial estimates in this thesis are proved using the assumption that we are dealing with a rectangle: one example of this is Lemma 2.8 which gives an important estimate for the reduction to the full energy to the local model. However, since the key part in that proof is that the Euclidean distance between two boundary points and the distance along the boundary are comparable, we expect that it should be possible to generalize it to polygons. On the other hand, some of the results in our analysis crucially depend on having a rectangle: one example is the uniqueness result for solution in Chapter 4, where our whole proof relies on having an angle of $\pi/2$. This is then further used to construct a recovery sequence which matches the lower bound for the energy in Chapter 7. It is not clear to us how to generalize this construction in the case of a general polygon, even though we think it should be possible. The case of a general Lipschitz domain remains unexplored and is likely even more difficult to tackle.

1.4 Notation

We gather here some elementary notation that is used throughout the thesis:

- We use a bold font to denote subsets of $\mathbb{R}^3$ and a normal font for subsets of $\mathbb{R}^2$. For example we will write $\boldsymbol{\Omega} \subset \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^2$.

- $\mathbf{m}_h$ denotes the magnetization, $\overline{\mathbf{m}}_h$ denotes the averaged magnetization and M_h the $\mathbb{S}^1$-valued replacement of the averaged magnetization.

- The Sobolev space $W^{s,p}(\Omega)$ for $p \geq 1, s > 0$ and Ω is defined as

  $$W^{s,p} := \{u \in L^p(\Omega) : u \in W^{\lfloor s \rfloor,p}(\Omega) \text{ such that } \sup_{|\alpha|=\lfloor s \rfloor} [D^\alpha u]_{s-\lfloor s \rfloor,p,\Omega} < \infty\}$$

  where $[\cdot]_{\theta,p,\Omega}$ denotes the Gagliardo-Slobodeckij seminorm defined for $\theta \in (0,1)$ as:

$$[u\cdot]_{\theta,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{\theta p+n}} dxdy\right)^{\frac{1}{p}}.$$

- We use the notation $H^p(\Omega)$ for the Sobolev space $W^{2,p}(\Omega)$, for $p \geq 1$.
- The full micromagnetic energy is defined for $\mathbf{m} \in H^1(\boldsymbol{\omega};\mathbb{S}^2)$ as:

$$E(\mathbf{m}) = d^2 \int_{\boldsymbol{\omega}} |\nabla \mathbf{m}|^2 d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla U|^2 d\mathbf{x} + Q\int_{\boldsymbol{\omega}} \varphi(\mathbf{m})\, d\mathbf{x} - \int_{\boldsymbol{\omega}} \mathbf{m}\cdot \mathbf{h}_{ext} d\mathbf{x}.$$

- The rescaled micromagnetic energy for a thin film of aspect ratio h is given by:

$$E_h(\mathbf{m}_h) = \frac{1}{h|\log\varepsilon|}\int_{\boldsymbol{\Omega}_h} |\boldsymbol{\nabla}\mathbf{m}_h|^2 d\mathbf{x} + \frac{1}{\eta^2 h|\log\varepsilon|}\int_{\mathbb{R}^3} |\boldsymbol{\nabla} U_h|^2 d\mathbf{x}.$$

- The reduced energy corresponding to the average magnetization is defined for $\overline{\mathbf{m}}_h \in H^1\left(\Omega;\overline{B}^2\right)$ (the unit disk in $\mathbb{R}^2$) as:

$$\overline{E}_h(\overline{\mathbf{m}}_h) := \frac{1}{|\log\varepsilon|}\left(\int_\Omega \left(|\nabla \overline{\mathbf{m}}_h|^2 + \frac{1}{\eta^2}\left(1-|\overline{m}_h|^2\right)\right)dx + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} (\overline{m}_h\cdot\nu)^2\, d\mathcal{H}^1\right),$$

  It can be extended to all of $H^1(\Omega;\mathbb{R}^2)$ by setting $\overline{E}_h(\mathbf{m}) = +\infty$ if $|\mathbf{m}| > 1$ on a set of positive measure.

- The energy $E_{\varepsilon,\eta}$ is defined for $u \in H^1(\Omega;\mathbb{R}^2)$ as:

$$E_{\varepsilon,\eta}(u) := \int_\Omega |\nabla u|^2 dx + \frac{1}{\eta^2}\int_\Omega \left(1-|u|^2\right)^2 dx + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} (u\cdot\nu)^2\, d\mathcal{H}^1.$$

Chapter 2

Reduction from the full to the local model

In this chapter, following Ignat and Kurzke [25], we show how, not only for $C^{1,1}$ domains as they do, but also for rectangles, we can replace the full micromagnetic energy with a local energy defined on a 2-dimensional domain which asymptotically does not change, replacing the magnetization with its average in the out-of-plane direction. In the next chapter we will show that we can further replace the averaged magnetization with a unit length vector, so that in the end we will be able to fully reduce the model (up to a rescaling) to the functionals

$$E_\varepsilon(u) := \int_\Omega |\nabla u|^2 + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \sin^2(u-g)\, d\mathcal{H}^1. \tag{2.1}$$

The first step is to perform a dimension reduction by averaging the magnetization in the out-of-plane direction[1]: for any thickness $h > 0$ we define the

[1]The intuition behind this is that since we consider thin films, the magnetization will not vary much in the out-of-plane direction, since this would cost a lot of energy. So we can replace it with its average without affecting the energy too much, provided we introduce a suitable replacement for the energy. In this chapter we will prove this rigorously.

averaged magnetization $\overline{\mathbf{m}}_h : \Omega \to \mathbb{R}^3$ as:

$$\overline{\mathbf{m}}_h (x) = \frac{1}{h} \int_{\Omega} \mathbf{m}_h (x, x_3)\, dx_3, \tag{2.2}$$

for every $x \in \Omega$. We observe that the averaging convexifies the unit-length constraints, i.e. $|\overline{\mathbf{m}}_h| \leq 1$ so that $\overline{\mathbf{m}}_h = (\overline{m}_h, \overline{m}_{h,3}) : \Omega \to \overline{B}^3$ (where $\overline{B}^3$ is the closed ball in $\mathbb{R}^3$). We introduce the following *reduced energy* corresponding to the average magnetization:

$$\overline{E}_h (\overline{\mathbf{m}}_h) := \frac{1}{|\log \varepsilon|} \left(\int_{\Omega} \left(|\nabla \overline{\mathbf{m}}_h|^2 + \frac{1}{\eta^2} \left(1 - |\overline{m}_h|^2\right) \right) dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (\overline{m}_h \cdot \nu)^2 \, d\mathcal{H}^1 \right), \tag{2.3}$$

where ν is the exterior normal vector field on $\partial\Omega$, and where the parameters are defined as in the introduction. We can extend the definition of $\overline{E}_h$ to the whole of $H^1(\Omega; \mathbb{R}^3)$ by setting $\overline{E}_h (\mathbf{m}) = +\infty$ if $|\mathbf{m}| > 1$ on a set of positive measure. We see that this energy penalizes spatial variations and pushes the average magnetization to be in-plane and tangential at the boundary.

In this chapter we will show that this substitution does not increase the energy asymptotically, in our regime. This has been done by Ignat and Kurzke [25] for domains with a $C^{1,1}$ boundary. We will extend their result for rectangles.

The second step (which we will carry out in the next chapter) consists in showing that we can replace the averaged magnetization $\mathbf{m}_h$ (which satisfies $\|\mathbf{m}_h\| \leq 1$) with a unit length magnetization M_h. Also in this case we will show that this does not change the energy much asymptotically, in our regime. The effect of this is to get rid of the second term in (2.3) (since the length of the vector m is now 1). So for this new family of magnetizations the energy will be:

$$|\log \varepsilon| E_\varepsilon (M_h) := \int_{\Omega} |\nabla M_h|^2 + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (M_h \cdot \nu)^2 \tag{2.4}$$

Now, by considering a lift u_ε of the magnetization (which exists thanks to [8, Lemma 4]), such that $M_h = e^{iu_\varepsilon}$ we obtain that the energy can be written as

$$E_\varepsilon\left(u_\varepsilon\right) := \int_\Omega |\nabla u|^2 + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \sin^2\left(u-g\right) d\mathcal{H}^1, \tag{2.5}$$

which is how we retrieve the functionals which are the object of our study. In the next two chapters we will explain how to do this. We start by reducing the model for the full energy to a 2-dimensional model. This – along with the reduction to $\mathbb{S}^1$-valued maps presented in the next section – shows how we go from the full micromagnetic energy to the scalar functionals that we investigate. We will consider, following [25], the following rescaled version of the micromagnetic energy, where we neglect the anisotropy term and assume there is no external field:

$$E_h\left(\mathbf{m}_h\right) := \frac{1}{h|\log\varepsilon|}\left(\int_{\mathbf{\Omega}_h} |\nabla \mathbf{m}_h|^2 d\mathbf{x} + \frac{1}{\eta^2}\int_{\mathbb{R}^3} |\nabla U_h|^2\right) \tag{2.6}$$

The main result of this chapter is the following theorem:

Theorem 2.1. *Let the energies E_h and $\overline{E}_h$ be defined as in (2.6) and (2.3) respectively. For a family $\mathbf{m}_h$ of magnetizations of uniformly bounded energy (i.e. $\limsup_{h\to 0} E_h\left(\mathbf{m}_h\right) < \infty$) we have the following estimate for the energy:*

$$E_h\left(\mathbf{m}_h\right) \geq \overline{E}_h\left(\overline{\mathbf{m}_h}\right) - \left(\overline{E}_h\left(\overline{\mathbf{m}_h}\right) + \sqrt{\frac{E_h\left(\mathbf{m}_h\right)}{|\log\varepsilon|}}\right) O\left(A\left(h\right)\right), \tag{2.7}$$

where $A\left(h\right)$ is defined as

$$A\left(h\right) := \frac{h}{\eta^2}\left(\frac{\log\frac{\eta^2}{h}}{|\log\varepsilon|} + 1\right) \tag{2.8}$$

The constants that are implied in the big-O notation only depend on the domain Ω. We have that in our regime $A\left(h\right) \ll 1$.

Before we prove the theorem (this will be done at the end of the chapter) we will need a few preliminary results. For now we show that $A(h) \ll 1$. By the definition of our regime (1.10) we have that

$$\frac{1}{|\log h|} \ll \varepsilon. \tag{2.9}$$

Now, since $a \ll b \ll 1$ implies $a|\log a| \ll b|\log b| \ll 1$ [2] we get that

$$\frac{\log|\log h|}{|\log h|} \ll \varepsilon|\log \varepsilon|. \tag{2.10}$$

Now we can compute from the definition of $A(h)$ and ε:

$$\begin{aligned} A(h) &= \frac{h}{\eta^2}\left(\frac{\log \frac{\eta^2}{h}}{|\log \varepsilon|} + 1\right) \\ &= \frac{1}{\varepsilon|\log h|}\left(\frac{\log(\varepsilon|\log h|) + |\log \varepsilon|}{|\log \varepsilon|}\right) \\ &= \frac{1}{\varepsilon|\log h|}\frac{\log \varepsilon + \log|\log h| + |\log \varepsilon|}{|\log \varepsilon|}. \end{aligned} \tag{2.11}$$

Now for $h \ll 1$ in the regime (1.10) we also have $\varepsilon \ll 1$ and so $\log \varepsilon = -|\log \varepsilon|$ therefore for h small enough we can rewrite the last expression as:

$$A(h) = \frac{1}{\varepsilon|\log h|}\frac{\log \varepsilon + \log|\log h| + |\log \varepsilon|}{|\log \varepsilon|} = \frac{1}{\varepsilon|\log \varepsilon|}\frac{\log|\log h|}{|\log h|}, \tag{2.12}$$

and now the conclusion that $A(h) \ll 1$ follows from (2.10). In order to show the estimate in (2.1) we compare the corresponding parts of the energy, i.e. the exchange energy and the stray-field energy, for the two energies. For the exchange energy we have by using Jensen's inequality that:

[2] We have that

$$\frac{a|\log a|}{b|\log b|} = \frac{a}{b}\frac{|\log \frac{a}{b} + \log b|}{|\log b|} \leq \frac{1}{|\log b|}\frac{a}{b}\left|\log \frac{a}{b}\right| + \frac{a}{b},$$

which tends to 0 since $a \ll b \ll 1$.

$$\int_{\Omega_h} |\nabla \overline{\mathbf{m}}_h|^2 dx \leq \frac{1}{h} \int_{\mathbf{\Omega}_h} |\nabla \mathbf{m}_h|^2 d\mathbf{x}. \tag{2.13}$$

To estimate the stray field energies we follow the strategy used by Ignat and Kurzke [25], which was in turn inspired by Kohn and Slastikov [32]. We first show that the stray-field energy for the magnetization $\mathbf{m}_h$ can be replaced by the corresponding term for the averaged magnetization $\overline{\mathbf{m}}_h$ without changing the energy too much. As a preliminary result we show that in the expression of the reduced energy (2.3) we can replace the second term with a different one, which is close to it asymptotically. We have the following lemma, which shows that the averaged magnetization is asymptotically close to the unit sphere:

Lemma 2.2. *Let $\mathbf{m}_h$ and $\overline{\mathbf{m}}_h$ be defined as above and satisfy the same assumptions of Theorem 2.1, i.e assume that $\limsup_{h\to 0} E_h(\mathbf{m}_h) < \infty$. Then we have the following estimate:*

$$0 \leq \frac{1}{\eta^2 |\log \varepsilon|} \left(\int_\Omega \left(1 - |\overline{m}_h|^2\right) dx - \int_\Omega \overline{m}_{h,3}^2 dx \right) = \sqrt{\frac{E_h(\mathbf{m}_h)}{|\log \varepsilon|}} O\left(\frac{h}{\eta^2}\right). \tag{2.14}$$

Proof. The proof is carried out in the same way as in [25], and only uses the Cauchy-Schwarz and Poincaré-Wirtinger inequalities. We repeat it here for the convenience of the reader: by Cauchy-Schwarz inequality we have that for $1 \leq j \leq 3$:

$$\int_{\mathbf{\Omega}_h} |m_{h,j}^2(x, x_3) - \overline{m}_{h,j}^2(x)| \leq 2\sqrt{h} \int_\Omega dx \left(\int_0^h |m_{h,j}(x, x_3) - \overline{m}_{h,j}(x)|^2 dx_3 \right)^{1/2}. \tag{2.15}$$

Now using the Poincaré-Wirtinger inequality we have that

$$
\begin{aligned}
\int_{\boldsymbol{\Omega}_h} |m_{h,j}^2(x,x_3) - \overline{m}_{h,j}^2(x)| &\leq 2\sqrt{h}\int_\Omega dx \left(\int_0^h |m_{h,j}(x,x_3) - \overline{m}_{h,j}(x)|^2 dx_3\right)^{1/2} \\
&\leq Ch^{3/2}\int_\Omega dx\left(\int_0^h |\partial_{x_3} m_{h,j}(x,x_3)|^2 dx_3\right)^{1/2} \\
&\leq Ch^{3/2}\left(\int_{\boldsymbol{\Omega}_h} |\partial_{x_3} m_{h,j}(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2}.
\end{aligned} \tag{2.16}
$$

Now summing over j and using the fact that $\mathbf{m}_h$ has length 1 we can deduce that

$$
\begin{aligned}
\int_\Omega \left(1 - |\overline{\mathbf{m}}_h|^2\right) dx &\leq \frac{1}{h}\sum_{j=1}^3 \int_{\boldsymbol{\Omega}_h} |m_{h,j}^2(x,x_3) - \overline{m}_{h,j}^2(x)| \\
&\leq Ch\left(|\log\varepsilon| E_h(\mathbf{m}_h)\right)^{1/2}.
\end{aligned} \tag{2.17}
$$

We now get the conclusion observing that $0 \leq 1 - |\overline{\mathbf{m}}_h|^2 = (1 - |\overline{m}_h|^2) - \overline{m}_{h,3}^2$. ☐

2.1 Asymptotic comparison for the stray-field energy

Lemma 2.3. *We have that*

$$
\frac{1}{\eta^2 h|\log\varepsilon|}\left|\int_{\mathbb{R}^3} |\nabla U_h|^2 d\mathbf{x} - \int_{\mathbb{R}^3} |\nabla \overline{U}_h|^2 d\mathbf{x}\right| \leq \frac{Ch}{\eta^2}\sqrt{\frac{E_h(\mathbf{m}_h)}{|\log\varepsilon|}}. \tag{2.18}
$$

Proof. This follows as in [25], which in turn uses a strategy by Kohn and Slastikov [32, Lemma 3]. The main ingredients are the Helmholtz projection and again the Poincaré-Wirtinger inequality. We have

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\nabla U_h|^2 d\mathbf{x} &\leq \int_{\mathbf{\Omega}_h} |\mathbf{m}_h|^2 d\mathbf{x} \leq Ch, \\
\int_{\mathbb{R}^3} |\nabla \overline{U_h}|^2 d\mathbf{x} &\leq \int_{\mathbf{\Omega}_h} |\overline{\mathbf{m}}_h|^2 d\mathbf{x} \leq Ch, \\
\int_{\mathbb{R}^3} |\nabla U_h - \nabla \overline{U_h}|^2 d\mathbf{x} &\leq \int_{\mathbf{\Omega}_h} |\mathbf{m}_h - \overline{\mathbf{m}}_h|^2 d\mathbf{x} \leq Ch^2 \int_{\boldsymbol{\Omega_h}} |\partial_{x_3} \mathbf{m}_h|^2,
\end{aligned} \tag{2.19}
$$

Using the inequality $|\|a\|^2 - \|b\|^2| \leq \left(2\|a-b\|^2\left(\|a\|^2 + \|b\|^2\right)\right)^{1/2}$ we get that:

$$
\begin{aligned}
\frac{1}{\eta^2 h |\log\varepsilon|} \left| \int_{\mathbb{R}^3} |\nabla U_h|^2 d\mathbf{x} - \int_{\mathbb{R}^3} |\nabla \overline{U_h}|^2 d\mathbf{x} \right| &\leq \frac{Ch}{\eta^2 |\log\varepsilon|} \left(\frac{1}{h} \int_{\mathbf{\Omega}_h} |\partial_{x_3} \mathbf{m}_h|^2 \right)^{1/2} \\
&\leq \frac{Ch}{\eta^2} \sqrt{\frac{E_h(\mathbf{m}_h)}{|\log\varepsilon|}}
\end{aligned} \tag{2.20}
$$

□

This result shows that we can focus on the stray-field energy corresponding to the average magnetization $\overline{\mathbf{m}}_h$. The next step is to compare this with the quantity

$$
\frac{1}{\eta^2} \int_\Omega \overline{m}_{h,3}^2 dx + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1. \tag{2.21}
$$

We have:

Proposition 2.4. *We have the following estimate:*

$$
\begin{aligned}
\frac{1}{|\log\varepsilon|} \left| \frac{1}{\eta^2 h} \int_{\mathbb{R}^3} |\nabla \overline{U}_h|^2 d\mathbf{x} - \frac{1}{\eta^2} \int_\Omega \overline{m}_{h,3}^2 dx - \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1 \right| &\leq \\
C \frac{h}{\eta^2} \left(1 + \frac{\log \frac{\eta^2}{h}}{|\log\varepsilon|} \right) \overline{E}_h\left(\overline{\mathbf{m}}_h\right),
\end{aligned} \tag{2.22}
$$

for a constant $C > 0$ only depending on Ω.

Proof. Since smooth functions are dense in $H^1\left(\Omega, \overline{B}^3\right)$ by the Meyers-Serrin theorem and since everything in (2.22) is continuous with respect to strong H^1 convergence, it is enough to prove our result for smooth[3] $\overline{\mathbf{m}}_h$. For such vector fields we can express the stray-field energy in the following way

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla \overline{U}_h|^2 d\mathbf{x} &\overset{(\dagger)}{=} \int_{\mathbf{\Omega}_h} \nabla \overline{U}_h \cdot \overline{\mathbf{m}}_h d\mathbf{x} \\ &\overset{(*)}{=} -\int_{\mathbf{\Omega}_h} \overline{U}_h \nabla \cdot \overline{\mathbf{m}}_h + \int_{\partial\mathbf{\Omega}_h} \overline{U}_h \left(\overline{\mathbf{m}}_h \cdot \boldsymbol{\nu}\right) d\mathcal{H}^2\left(\mathbf{x}\right),\end{aligned} \tag{2.23}$$

where $\boldsymbol{\nu}$ is the outer normal on $\partial\mathbf{\Omega}_h$ and for ($\dagger$) we have used Maxwell's equation and in ($*$) we used integration by parts. By Proposition 25 in [25] we can express the stray-field energy in the following way:

$$4\pi \overline{U}_h\left(\mathbf{x}\right) = -\int_{\mathbf{\Omega}_h} \frac{1}{|\mathbf{x}-\mathbf{y}|} \nabla \cdot \overline{m}_h\left(y\right) d\mathbf{y} + \int_{\partial\mathbf{\Omega}_h} \frac{1}{|\mathbf{x}-\mathbf{y}|} \left(\overline{\mathbf{m}}_h \cdot \boldsymbol{\nu}\right)\left(\mathbf{y}\right) d\mathcal{H}^2\left(\mathbf{y}\right),$$

where in the first term we can write $\nabla \cdot \overline{m}_h\left(y\right)$ instead of $\boldsymbol{\nabla} \cdot \overline{\mathbf{m}}_h\left(\mathbf{y}\right)$ because $\overline{m}_h$ by definition does not depend on the third component of $\mathbf{y}$, but only on $y \in \Omega$.

Now we can express the stray-field energy for the averaged magnetization as:

$$4\pi \int_{\mathbb{R}^3} |\boldsymbol{\nabla} \overline{U}_h|^2 d\mathbf{x} = \mathcal{A} + 2\mathcal{B} + \mathcal{C}, \tag{2.24}$$

where

- $\mathcal{A}$ is the "bulk-bulk term":

$$\mathcal{A} = \int_0^h \int_0^h \int_\Omega \int_\Omega \frac{\nabla \cdot \overline{m}_h\left(x\right) \nabla \cdot \overline{m}_h\left(y\right)}{\sqrt{|x-y|^2 + \left(x_3 - y_3\right)^2}} d\mathbf{x} d\mathbf{y}, \tag{2.25}$$

[3]This assumptions allows us to not worry about the well-definedness of some integrals, but it might not be necessary to assume this. The estimate we prove holds for all H^1 functions.

- $\mathcal{B}$ is the "bulk-boundary term":

$$\mathcal{B} = \int_{\mathbf{\Omega}_h} \int_{\partial \mathbf{\Omega}_h} \frac{\nabla \cdot \overline{m}_h (x) \left(\overline{\mathbf{m}}_h \cdot \boldsymbol{\nu} \right) (\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{x} d\mathbf{y}. \tag{2.26}$$

- $\mathcal{C}$ is the "boundary-boundary term":

$$\mathcal{C} = \int_{\partial \mathbf{\Omega}_h} \int_{\partial \mathbf{\Omega}_h} \frac{\left(\overline{\mathbf{m}}_h \cdot \boldsymbol{\nu} \right) (\mathbf{x}) \left(\overline{\mathbf{m}}_h \cdot \boldsymbol{\nu} \right) (\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{x} d\mathbf{y}. \tag{2.27}$$

2.1.1 Estimates for $\mathcal{A}$ and $\mathcal{B}$

This follows verbatim as in the case of smooth domains: to estimate $\mathcal{A}$ and $\mathcal{B}$ we can use the generalized Young inequality (see [32, Lemma 1]) and the estimate given in [32, Lemma 2][4] respectively:

$$\begin{aligned} |\mathcal{A}| &\leq h^2 \int_{\Omega} \int_{\Omega} \frac{|\nabla \cdot \overline{m}_h (x)| |\nabla \cdot \overline{m}_h (y)|}{|x - y|} dx dy \\ &\leq C h^2 \int_{\Omega} |\nabla \cdot \overline{m}_h|^2 dx \leq C h^2 |\log \varepsilon| \overline{E}_h \left(\overline{\mathbf{m}}_h \right). \end{aligned} \tag{2.28}$$

For $\mathcal{B}$ we first notice that the integrals corresponding to the top and bottom boundary $\partial \mathbf{\Omega}_h$ cancel out after integration, and we are left with

[4] An alternative short proof of this result can be found in [25]. We report it here. Let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$: for $y \in \partial\Omega$ define $F(y) = \int_\Omega \frac{f(x)}{|x-y|}$. Then Hölder's inequality implies that $F^2(y) \leq \int_\Omega \frac{1}{|x-y|^{3/2}} dx \int_\Omega \frac{f^2(y)}{|x-y|^{1/2}} dx$ and so $\int_{\partial\Omega} F^2(y) \, d\mathcal{H}^1(y) \leq c(\Omega) \|f\|^2_{L^2(\Omega)} \sup_{x\in\Omega} \int_{\partial\Omega} \frac{1}{|x-y|^{1/2}} d\mathcal{H}^1(y) \leq C(\Omega) \|f\|^2_{L^2(\Omega)}$. Then the claim of [32, Lemma 2] follows using the Cauchy-Schwarz inequality.

$$
\begin{aligned}
|\mathcal{B}| &= \left| \int_0^h \int_0^h \int_\Omega \int_{\partial\Omega} \frac{\nabla \cdot \overline{m}_h(x)\left(\overline{m}_h \cdot \nu\right)(y)}{\sqrt{|x-y|^2 + (x_3 - y_3)^2}} d\mathbf{x} d\mathbf{y} \right| \\
&\leq h^2 \int_\Omega \int_{\partial\Omega} \frac{|\nabla \cdot \overline{m}_h(x)||\left(\overline{m}_h \cdot \nu\right)(y)|}{|x-y|} \\
&\leq Ch^2 \|\nabla \cdot \overline{m}_h\|_{L^2(\Omega)} \|\overline{m}_h \cdot \nu\|_{L^2(\partial\Omega)} \\
&\leq Ch^2 \varepsilon^{1/2} |\log \varepsilon| \overline{E}_h(\overline{\mathbf{m}}_h)
\end{aligned}
\tag{2.29}
$$

2.1.2 Estimate for $\mathcal{C}$

To estimate $\mathcal{C}$ we write first $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ where

$$
\begin{aligned}
\mathcal{C}_1 &= 4\pi h \int_\Omega \int_\Omega \overline{m}_{h,3}(x)\, \overline{m}_{h,3}(y)\, \Gamma_h(x-y)\, dx dy \\
\mathcal{C}_2 &= \int_0^h \int_0^h \int_{\partial\Omega} \int_{\partial\Omega} \frac{\left(\overline{m}_h \cdot \nu\right)(x)\left(\overline{m}_h \cdot \nu\right)(y)}{|\mathbf{x} - \mathbf{y}|} d\mathbf{x} d\mathbf{y},
\end{aligned}
\tag{2.30}
$$

where Γ_h is defined for all $x \in \mathbb{R}^2$ as:

$$
\Gamma_h(x) := \frac{1}{2\pi h}\left(\frac{1}{|x|} - \frac{1}{\sqrt{|x|^2 + h^2}}\right) \tag{2.31}
$$

The estimate for $\mathcal{C}_1$ works for a rectangle in the same as it does for a $C^{1,1}$ domain, and the proof is carried out as in [25]. Namely we have:

Lemma 2.5. *We have the following estimate for $\mathcal{C}_1$:*

$$
\frac{1}{\eta^2 |\log \varepsilon|}\left|\frac{\mathcal{C}_1}{4\pi h} - \int_\Omega \overline{m}_{h,3}^2(x)\, dx\right| \leq C \frac{h}{\eta^2}\left(\frac{\log \frac{\eta^2}{h}}{|\log \varepsilon|} + 1\right) \overline{E}_h(\overline{\mathbf{m}}_h) \tag{2.32}
$$

Proof. The proof is the same as in [25]. Since $\operatorname{diam}(\Omega) = 1$ we can use the fact that $\Gamma_h(x) = \frac{h}{2\pi|x|^2}\rho_h(|x|)$ for $x \in B^2 \subset \mathbb{R}$ where ρ_h is defined for $r \geq 0$ as

$$
\rho_h(r) = \frac{r}{\left(r + \sqrt{r^2 + h^2}\right)\sqrt{r^2 + h^2}} \mathbf{1}_{0 \leq r \leq 1}(r),
$$

where $\mathbf{1}$ denotes the characteristic function. We observe that ρ_h is bounded in $L^1(\mathbb{R}^2)$. Indeed we have (observing that $\rho_h(r) \leq \frac{r}{2r^2}\mathbf{1}_{0\leq r\leq 1}(r) = \frac{1}{2r}\mathbf{1}_{0\leq r\leq 1}(r)$ and using polar coordinates):

$$\int_{\mathbb{R}^2} \rho_h(|x|)\, dx \leq \pi.$$

For every $0 < R \leq 1$ we can compute

$$\begin{aligned}\int_{B_R(0)} \Gamma_h(x)\, dx &= h\int_0^R \frac{dr}{\left(r+\sqrt{r^2+h^2}\right)\sqrt{r^2+h^2}} \\ &= \int_0^{\frac{R}{h}} \frac{ds}{\left(s+\sqrt{s^2+1}\right)\sqrt{s^2+1}} = 1 - \frac{1}{\frac{R}{h}+\sqrt{1+\left(\frac{R}{h}\right)^2}} \leq 1.\end{aligned}$$

In particular we get, for $0 < R \leq 1$, that

$$0 \leq 1 - \int_{B_R(0)} \Gamma_h(x)\, dx \leq \frac{h}{R}. \tag{2.33}$$

We now define, for all $r \leq r_0$, where $r_0 > 0$ is fixed and small, the set Ω_r of points that have distance less than r from the boundary (whose complement is a rectangle) as

$$\Omega_r = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < r\}.$$

For $x, y \in \Omega$ we can write

$$2\overline{m}_{h,3}(x)\,\overline{m}_{h,3}(y) = \overline{m}_{h,3}(x)^2 + \overline{m}_{h,3}(y)^2 - \left(\overline{m}_{h,3}(x) - \overline{m}_{h,3}(y)\right)^2.$$

We can now rewrite $\mathcal{C}_1$ as

$$\mathcal{C}_1 = -\mathcal{E}_1 + \mathcal{E}_2,$$

where

$$\mathcal{E}_1 = h^2 \int_\Omega \int_\Omega \frac{\left(\overline{m}_{h,3}(x) - \overline{m}_{h,3}(y)\right)^2}{|x-y|^2} \rho_h(|x-y|)\, dxdy$$
$$\mathcal{E}_2 = 4\pi h \int_\Omega \int_\Omega \overline{m}_{h,3}^2(x)\, \Gamma_h(|x-y|)\, dxdxy,$$

and where we have used that $\Gamma_h(x) = \frac{h}{2\pi|x|^2}\rho_h(|x|)$.
Let $T : H^1(\Omega) \to H^1(\mathbb{R}^2)$ be a bounded linear extension operator, which exists since Ω is Lipschitz. Then we can estimate $\mathcal{E}_1$ as follows:

$$\begin{aligned}
0 \le \mathcal{E}_1 &= h^2 \int_\Omega \int_\Omega \frac{\left(\overline{m}_{h,3}(x) - \overline{m}_{h,3}(y)\right)^2}{|x-y|^2} \rho_h(|x-y|)\, dxdy \\
&\le h^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_0^1 \Big| \nabla\left[T\left(\overline{m}_{h,3}\right)\right](x + s(y-x)) \Big|^2 \rho_h(|x-y|)\, dxdyds \\
&\overset{(\dagger)}{\le} h^2 \int_{\mathbb{R}^2} |\nabla\left[T\left(\overline{m}_{h,3}\right)\right](x)|^2 dx \int_{\mathbb{R}^2} \rho_h(|y|)\, dy \\
&\le Ch^2 \left(\int_\Omega |\nabla \overline{m}_{h,3}(x)|^2 dx + \overline{m}_{h,3}^2 \right) \le Ch^2 |\log \varepsilon| \overline{E}_h(\overline{\mathbf{m}}_h),
\end{aligned}$$

where we used $\overline{m}_{h,3}^2 \le 1 - |\overline{m}_h|^2, \eta \le 1$ to estimate $\overline{m}_{h,3}^2 \le \frac{1}{\eta^2}(1 - |\overline{m}_h|^2)$. Furthermore we have obtained the estimate ($\dagger$) as follows (and using the fact that both factors in the integrand are positive):

$$\begin{aligned}
& h^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_0^1 \Big| \nabla\left[T\left(\overline{m}_{h,3}\right)\right](x + s(y-x)) \Big|^2 \rho_h(|x-y|)\, dxdyds \\
&\overset{y-x=z}{\le} h^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_0^1 \Big| \nabla\left[T\left(\overline{m}_{h,3}\right)\right](x + sz) \Big|^2 \rho_h(|z|)\, dxdsdz \\
&= h^2 \int_{\mathbb{R}^2} \rho_h(|z|) \left[\int_{\mathbb{R}^2} \int_0^1 \Big| \nabla\left[T\left(\overline{m}_{h,3}\right)\right](x + sz) \Big|^2 dsdx \right] dz \\
&\overset{(*)}{\le} h^2 \int_{\mathbb{R}^2} \rho_h(|z|) \int_{\mathbb{R}^2} \Big| \nabla\left[T\left(\overline{m}_{h,3}\right)\right](y) \Big|^2 dydz,
\end{aligned}$$

from which the desired conclusion follows – notice that in $(*)$ we have used the mean value theorem to write, for each fixed z, $\int_0^1 \Big| \nabla\left[T\left(\overline{m}_{h,3}\right)\right](x+sz) \Big|^2 ds =$

$\left|\nabla\left[T\left(\overline{m}_{h,3}\right)\right](x+s_z z)\right|^2$, for $s_z \in (0,1)$, from which the conclusion follows, since for all fixed z we have $\int_{\mathbb{R}^2}\left|\nabla\left[T\left(\overline{m}_{h,3}\right)\right](x+s_z z)\right|^2 dx = \int_{\mathbb{R}^2}\left|\nabla\left[T\left(\overline{m}_{h,3}\right)\right](y)\right|^2 dy$. Now to the estimate of $\mathcal{E}_2$. Since $\eta \to 0$ we can assume that (in our regime (1.10)) $2h \leq \eta^2 \leq \frac{r_0}{2}$ we have, by decomposing the domain as $\Omega = \Omega_h \cup (\Omega_{\eta^2} \setminus \Omega_h) \cup (\Omega \setminus \Omega_{\eta^2})$ and using the fact that $\overline{m}_{h,3}^2 \leq 1 - |\overline{m}_h|^2 \leq 1$:

$$
\begin{aligned}
\left|\frac{\mathcal{E}_2}{4\pi h} - \int_\Omega \overline{m}_{h,3}^2 dx\right| &= \int_\Omega \overline{m}_{h,3}^2(x)\left(1 - \int_\Omega \Gamma_h\left(|x-y|dy\right)\right)dx \\
&\overset{\dagger}{\leq} \int_{\Omega_h} 1dx + \int_{\Omega_{\eta^2}\setminus\Omega_h} \frac{h}{\operatorname{dist}(x,\partial\Omega)}dx + \frac{h}{\eta^2}\int_{\Omega\setminus\Omega_{\eta^2}}\left(1 - |\overline{m}_h|^2(x)\right)dx \\
&\leq C\left(h + h\int_h^{\eta^2}\frac{dr}{r}\right) + h|\log\varepsilon|\overline{E}_h(\overline{\mathbf{m}}_h) \\
&\leq Ch\left(\log\frac{\eta^2}{h} + |\log\varepsilon|\right)\overline{E}_h(\overline{\mathbf{m}}_h),
\end{aligned}
$$

where C only depends on Ω and where in the inequality marked with $\dagger$ we have used (2.33) along with the fact that $1 - \int_\Omega \Gamma_h(|x-y|)\,dy \leq 1 - \int_{B(0,\operatorname{dist}(x,\partial\Omega))}\Gamma_h(|z|)\,dz$ for $x \in \Omega_{\eta^2}\setminus\Omega_h$ and that $1 - \int_\Omega \Gamma_h(|x-y|)\,dy \leq 1 - \int_{B(0,\eta^2)}\Gamma_h(|z|)\,dz$ for $x \in \Omega \setminus \Omega_{\eta^2}$. We have also used he fact that $\overline{E}_h(\overline{\mathbf{m}}_h) \geq 2\pi - o(h)$ as $h \to 0$, see Lemma 2.11 below, to estimate $1 \leq \overline{E}_h(\overline{\mathbf{m}}_h)$ in the first term and so in particular we get $Ch \leq Ch|\log\varepsilon|\overline{E}_h(\overline{\mathbf{m}}_h)$. Now from the estimate for $\mathcal{E}_1$ we get $\frac{1}{\eta^2|\log\varepsilon|}\left|\frac{\mathcal{E}_1}{4\pi h}\right| \leq \frac{Ch}{\eta^2}\overline{E}_h(\overline{\mathbf{m}}_h)$ and from that for $\mathcal{E}_2$ we get $\frac{1}{\eta^2|\log\varepsilon|}\left|\frac{\mathcal{E}_2}{4\pi h} - \int_\Omega \overline{m}_{h,3}^2 dx\right| \leq \frac{Ch}{\eta^2}\left(\frac{\log\frac{\eta^2}{h}}{|\log\varepsilon|} + 1\right)\overline{E}_h(\overline{\mathbf{m}}_h)$, from which we get the conclusion. □

It remains to estimate $\mathcal{C}_2$. In this case the proof in [25] does not work, because it uses in a crucial way the hypothesis that the domain is $C^{1,1}$. We can still prove an analogous result for the rectangle but we have to replace some steps of the proof. We state the result as

Lemma 2.6. *We have the following estimate for $\mathcal{C}_2$:*

$$\frac{1}{|\log \varepsilon|}\left|\frac{\mathcal{C}_2}{4\pi\eta^2 h} - \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1\right| \leq C\frac{h}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right). \tag{2.34}$$

We need some preliminary results before we can prove this. The approach is the same as in [25], where we split $\mathcal{C}_2$ as

$$\frac{\mathcal{C}_2}{4\pi\eta^2 h|\log \varepsilon|} = \frac{\mathcal{G}_1 + \mathcal{G}_2}{4\pi}, \tag{2.35}$$

where

$$\begin{aligned}
\mathcal{G}_1 &:= \frac{h}{\eta^2|\log \varepsilon|}\int_{\partial\Omega}\int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 (x)\, K_h\left(x-y\right) dxdy \\
\mathcal{G}_2 &:= \frac{h}{\eta^2|\log \varepsilon|}\int_0^1\int_0^1\int_{\partial\Omega}\int_{\partial\Omega} \frac{\left(\overline{m}_h \cdot \nu\right)(x)\left(\left(\overline{m}_h \cdot \nu\right)(x) - \left(\overline{m}_h \cdot \nu\right)(y)\right)}{\sqrt{|x-y|^2 + h^2\left(s-t\right)^2}} dxdydsdt,
\end{aligned} \tag{2.36}$$

and K_h is defined for all $z \in \mathbb{R}^2$ as follows:

$$K_h\left(z\right) = \int_0^1\int_0^1 \frac{1}{\sqrt{|z|^2 + h^2\left(s-t\right)^2}} dsdt. \tag{2.37}$$

We then estimate $\frac{\mathcal{G}_1}{4\pi} - \frac{1}{2\pi\varepsilon|\log\varepsilon|}\int_{\partial\Omega}\left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1$ and $\mathcal{G}_2$ separately.

Estimate for $\mathcal{G}_1$

We have the following result:

Lemma 2.7. *We have the following estimate for $\mathcal{G}_1$:*

$$\left|\frac{\mathcal{G}_1}{4\pi} - \frac{1}{2\pi\varepsilon|\log \varepsilon|}\int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1\right| \ll \frac{Ch}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right). \tag{2.38}$$

Proof. We have:

$$
\begin{aligned}
&\left| \frac{\mathcal{G}_1}{4\pi} - \frac{1}{2\pi\varepsilon|\log\varepsilon|} \int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1 \right| \\
&\leq \frac{1}{4\pi\varepsilon|\log\varepsilon|} \int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1 \left\| 2 - \frac{1}{|\log h|} \int_{\partial\Omega} K_h\left(x-y\right) dy \right\|_{L^\infty(\partial\Omega)} \\
&\leq \frac{C}{|\log h|} \overline{E}_h\left(\overline{\mathbf{m}}_h\right) \ll \frac{h}{\eta^2} \overline{E}_h\left(\overline{\mathbf{m}}_h\right).
\end{aligned}
$$

□

In the proof we have used the following result:

Lemma 2.8. *We have, for all $h < h_0$, for some $h_0 > 0$:*

$$\sup_{x\in\partial\Omega} \left| \frac{1}{|\log h|} \int_{\partial\Omega} K_h\left(x-y\right) dy - 2 \right| \leq \frac{C}{|\log h|}. \tag{2.39}$$

Proof. We follow the proof in [25, Lemma 7]. We have that (see [25] for the proof) by a calculation that

$$\int_{\partial\Omega} K_h\left(x-y\right) dy = \frac{2}{h} \int_{\partial\Omega} f\left(\frac{|x-y|}{h}\right) dy, \tag{2.40}$$

where

$$f\left(t\right) := \operatorname{arsinh}\left(\frac{1}{t}\right) - \frac{1}{t+\sqrt{1+t^2}}, \qquad t > 0. \tag{2.41}$$

We list a few properties of f that will be useful:

1. $f\left(t\right) > 0$ and $f'\left(t\right) < 0$ for all $t > 0$, i.e. f is positive and decreasing.

2. We have the following asymptotic behaviour of f at 0 and $+\infty$ respectively:

$$\lim_{t\to 0} \frac{f\left(t\right)}{\log\frac{1}{t}} = 1 \quad \text{and} \quad \left| f\left(t\right) - \frac{1}{2t} \right| \leq \frac{C}{t^3} \text{ as } t \to \infty. \tag{2.42}$$

3. We have that

$$\left|\int_0^t f(s)\,ds - \frac{\log t}{2}\right| \le C \text{ as } t \to \infty. \tag{2.43}$$

To compute the integral on the right-hand side of (2.40) for a given $x \in \partial\Omega$ we follow a similar approach to Ignat and Kurzke, although the presence of the corner makes things harder to compute. We choose a $\kappa > 0$ small enough (and in any case such that $2\kappa < \min\{a, b\}$ where a and b are the side-lengths. We take an arc-length parametrization $\varphi : [0, 2(a+b)) \to \partial\Omega$ of $\partial\Omega$ such that $\varphi(0) = x$.

Then we split the integral in two as

$$\underbrace{\frac{2}{h}\int_{\partial\Omega} f\left(\frac{|x-y|}{h}\right) dy}_{I(h,x)} = \underbrace{\frac{2}{h}\int_{-\kappa}^{\kappa} f\left(\frac{|\varphi(s)-x|}{h}\right) dy}_{I_1(h,x)} + \underbrace{\frac{2}{h}\int_{\kappa}^{2(a+b)-\kappa} f\left(\frac{|\varphi(s)-x|}{h}\right) dy}_{I_2(h,x)}.$$

To estimate I_2 we observe f is positive so we obviously have $I_2 \ge 0$ and furthermore using that f is monotone decreasing and that $|\varphi(s) - x| \ge \kappa/\sqrt{2}$ for $s \in (\kappa, 2(a+b) - \kappa)$ we get

$$I_2(x,h) \le \frac{2}{h}\int_{\kappa}^{2(a+b)-\kappa} f\left(\frac{\kappa}{\sqrt{2}h}\right) ds = \frac{4\sqrt{2}(a+b-\kappa)}{\kappa}\left(\frac{\kappa}{\sqrt{2}h} f\left(\frac{\kappa}{\sqrt{2}h}\right)\right) \le C, \tag{2.44}$$

when $h < h_0$ for some $h_0 \in (0,1)$. This holds since $tf(t) \to \frac{1}{2}$ as $t \to \infty$, and so $|tf(t)| \le C$, for some $C > 0$ when $t > t_0$, for some $t_0 > 0$. It remains to estimate $I_1(x,h)$, for all $x \in \partial\Omega$. We distinguish two different cases:

- Consider first the case where the distance of x from any vertex is greater than κ, or when x is one of the vertices. Then we have:

$$\frac{2}{h}\int_{-\kappa}^{\kappa} f\left(\frac{|\varphi(s)-x|}{h}\right) ds = \frac{4}{h}\int_0^{\kappa} f\left(\frac{s}{h}\right) ds = 4\int_0^{\frac{\kappa}{h}} f(t)\,dt \tag{2.45}$$

and we have from (2.43) that this can be bounded as

$$2|\log h| - C \leq 4 \int_0^{\frac{\kappa}{h}} f(t)\, dt \leq 2|\log h| + C, \tag{2.46}$$

where C is a constant that only depends on κ.

- Consider now the case in which x is not one of the vertices, but its distance from one of the vertices is less than κ. Without loss of generality we can assume that all the points $\varphi(s)$ for $s \in (-\kappa, 0)$ lie on one side of the rectangle, and we can handle this case as previously, obtaining:

$$|\log h| - C \leq \frac{2}{h} \int_{-\kappa}^{0} f\left(\frac{|\varphi(s) - x|}{h}\right) ds \leq |\log h| + C. \tag{2.47}$$

To compute $\frac{2}{h}\int_0^{\kappa} f\left(\frac{s}{h}\right) ds$ we proceed as follows. It is clear that this can be rewritten as (where by a slight abuse of notation we denote by x the distance of x from the vertex)

$$\frac{2}{h}\int_0^{\kappa} f\left(\frac{|\varphi(s) - x|}{h}\right) ds = \frac{2}{h}\int_0^{x} f\left(\frac{s}{h}\right) ds + \frac{2}{h}\int_0^{\kappa - x} f\left(\frac{\sqrt{x^2+s^2}}{h}\right) ds. \tag{2.48}$$

We will obtain the desired estimate if we can show that

$$\left|\frac{1}{h}\int_0^{\kappa - x} f\left(\frac{\sqrt{x^2+s^2}}{h}\right) ds + \frac{1}{h}\int_0^{x} f\left(\frac{s}{h}\right) ds - \frac{1}{h}\int_0^{\kappa} f\left(\frac{s}{h}\right) ds\right| \leq C, \tag{2.49}$$

for all $h \in (0,1)$ and all $x \in [0,\kappa]$, for a constant independent of x and h. This is obviously true when $x \in \{0,\kappa\}$, because in that case the LHS is 0. So in the following we can assume that $x \in (0,\kappa)$. We do this by showing that for all choices of x and h small enough (e.g. $x, h \in (0,1)$) we can bound the expression in (2.49) by a constant which does not depend on x or h, but only on κ. We distinguish two cases:

– We first assume that $x \geq h$. We introduce the following notation to make the proof easier to follow:

$$\begin{aligned} A &:= \frac{1}{h}\int_0^{\kappa - x} f\left(\frac{\sqrt{x^2+s^2}}{h}\right) ds, \\ B &:= \frac{1}{h}\int_0^{x} f\left(\frac{s}{h}\right) ds - \frac{1}{h}\int_0^{\kappa} f\left(\frac{s}{h}\right) ds. \end{aligned} \tag{2.50}$$

We observe that $A > 0$ and $B < 0$. We first show the proof under the assumption that $A + B \geq 0$. We are looking for an upper bound for $|A + B| = A + B$. The term A can be estimated by

$$A \leq C - \frac{1}{2}\log x. \tag{2.51}$$

Indeed we observe that for f the following estimate holds (we only need it for an argument ≥ 1 because that is always the case when we apply it below):

$$f(z) \leq \frac{1}{2z}, \qquad \text{for all } z \geq 1. \tag{2.52}$$

We give the proof in the following lemma:

Lemma 2.9. *For $z \geq 1$ we have:*

$$f(z) \leq \frac{1}{2z}$$

Proof. We prove this showing that the inequality is true for $z = 1$, that the function $f(z) - \frac{1}{2z}$ is strictly increasing and that $\lim_{z\to\infty} f(z) - \frac{1}{2z} = 0$.

The latter is true because of the second part of (2.42). We have $f(1) - \frac{1}{2} = \operatorname{arsinh}(1) - \frac{1}{1+\sqrt{2}} - \frac{1}{2} < 0$. So we only need to show that $f(z) - \frac{1}{2z}$ is strictly increasing. We compute the derivative and we get:

$$\left(f(z) - \frac{1}{2z}\right)' = 1 - \frac{\sqrt{1+z^2}}{z} + \frac{1}{2z^2} = \frac{2z^2 - 2z\sqrt{z^2+1} + 1}{2z^2}$$

It is thus enough to prove that

$$2z^2 - 2z\sqrt{z^2+1} + 1 > 0.$$

We have from the inequality $1 + a < \left(1 + \frac{a}{2}\right)^2$ that:

$$1 + \frac{1}{z^2} < \left(1 + \frac{1}{2z^2}\right)^2,$$

from which we get

$$\sqrt{1 + \frac{1}{z^2}} < 1 + \frac{1}{2z^2}.$$

If we multiply both sides by $2z^2$ we get $2z\sqrt{1+z^2} < 2z^2 + 1$, from which it follows that $2z^2 - 2z\sqrt{z^2+1} + 1 > 0$ for all z. This concludes the proof of (2.52).

□

We can now continue the proof of Lemma 2.8 and estimate as follows[5]:

[5]Here we use the fact that the function $g(s) := \log\left(\sqrt{s^2+x^2} + s\right)$ is a primitive of $\frac{1}{\sqrt{s^2+x^2}}$, as can be verified easily by a simple calculation, computing the derivative of g with respect to s.

$$\frac{1}{h}\int_0^{\kappa-x} f\left(\frac{\sqrt{s^2+x^2}}{h}\right) ds \leq \frac{1}{h}\int_0^{\kappa-x} \frac{1}{2}\frac{h}{\sqrt{s^2+x^2}} ds$$
$$= \frac{1}{2}\log\left(\sqrt{(\kappa-x)^2+x^2}+\kappa-x\right) - \frac{1}{2}\log x. \tag{2.53}$$

Let $q(x) := \log\left(\sqrt{(\kappa-x)^2+x^2}+\kappa-x\right)$ and observe that q is a continuous function on $[0,\kappa]$ and that $q(0) = \log(2\kappa), q(\kappa) = \log\kappa$. Thus q is bounded on $[0,\kappa]$, since it is a continuous function on a compact set. From this we obtain (2.51).

For B we use the estimate (2.43) to conclude that for functions $C_1(h)$ and $C_2(h)$ - note that C_2 depends on our choice of κ, but this is not an issue, since κ is fixed - which are uniformly bounded in h (since $x \geq h$, so $x/h \geq 1$, and since κ is fixed) we have:

$$\frac{1}{h}\int_0^x f\left(\frac{s}{h}\right) ds - \frac{1}{h}\int_0^\kappa f\left(\frac{s}{h}\right) ds = \int_0^{\frac{x}{h}} f(s)\, ds - \int_0^{\frac{\kappa}{h}} f(s)\, ds$$
$$= \frac{1}{2}\log\frac{x}{h} + C_1(h) - \left(\frac{1}{2}\log\frac{\kappa}{h} + C_2(h)\right)$$
$$= \frac{1}{2}\log\frac{x}{\kappa} + C(h), \tag{2.54}$$

where $C(h)$ is uniformly bounded in h. Putting everything together we obtain that for a constant $C > 0$:

$$|A+B| \leq C - \frac{1}{2}\log(x) + \frac{1}{2}\log\frac{x}{\kappa} = C + \frac{1}{2}\log\frac{1}{\kappa}. \tag{2.55}$$

Now we assume that $A + B \leq 0$, so that $|A + B| = -A - B$. For $-B$ the estimate is done exactly as before, only the sign will now be reversed. To find an upper bound on $-A - B$ is then enough to find an upper bound on $-A$. We observe that there exists a constant $C > 0$ such that:

$$-f(z) + \frac{1}{2z} - \frac{C}{z^3} < 0 \text{ for all } z \geq 1. \tag{2.56}$$

In fact we have from (2.42) that there exists $z_0 > 0$ and a constant $C_0 > 0$ such that for all $z > z_0$

$$\left| f(z) - \frac{1}{2z} \right| \leq \frac{C_0}{z^3}. \tag{2.57}$$

Since we have $f(z) \leq \frac{1}{2z}$ for all $z > 0$, we derive that for all $z > z_0$:

$$-f(z) + \frac{1}{2z} - \frac{C_0}{z^3} < 0. \tag{2.58}$$

We now want to observe that (by making C large if necessary) this inequality holds for all $z \geq 1$. Let (for a C to be chosen later)

$$g(z) = -f(z) + \frac{1}{2z} - \frac{C}{z^3}. \tag{2.59}$$

Then we have that

$$g'(z) = -f'(z) - \frac{1}{2z^2} + \frac{3C}{z^4}. \tag{2.60}$$

Now notice that $-f'(z) > 0$ and that $-\frac{1}{2z^2} + \frac{3C}{z^4} > 0$ if $z \leq \sqrt{6C}$, which is true on $[1, z_0]$ if we choose $\sqrt{6C} > z_0$. So for C chosen large enough we conclude that g is an increasing function, so $g(z) < g(z_0)$ for all $z \in [1, z_0]$. This, combined with (2.58) (which remains true if we increase C_0) we obtain the desired estimate for f. Now we can proceed to estimate $-A$ as follows (where we use that $x \geq h$ to get that the argument of f in the integral is greater that 1, so that we can apply our estimate):

$$
\begin{aligned}
-\frac{1}{h}\int_0^{\kappa - x} f\left(\frac{\sqrt{s^2+x^2}}{h}\right) ds \leq & -\frac{1}{h}\int_0^{\kappa-x}\frac{1}{2}\frac{h}{\sqrt{s^2+x^2}}ds + \frac{C}{h}\int_0^{\kappa-x}\frac{h^3}{\left(x^2+s^2\right)^{3/2}}ds \\
&= -\frac{1}{2}\log\left(\sqrt{\left(\kappa - x\right)^2 + x^2} + \kappa - x\right) + \frac{1}{2}\log x \\
&+ \frac{Ch^2}{x^2}\int_0^{\frac{\kappa-x}{x}}\frac{1}{\left(1+y^2\right)^{3/2}}dy \\
&\leq -\frac{1}{2}\log\left(\sqrt{\left(\kappa - x\right)^2 + x^2} + \kappa - x\right) + \frac{1}{2}\log x \\
&+ C\int_0^{\infty}\frac{1}{\left(1+y^2\right)^{3/2}}dy \\
&\leq \frac{1}{2}\log x + C.
\end{aligned}
\tag{2.61}
$$

Now we get the estimate for $-A - B$ in the same way as above.

- We now assume that $x \leq h$. In this case we need different estimates. We estimate $\frac{1}{h}\int_0^x f\left(\frac{s}{h}\right) ds$ using (2.43) as

$$
\frac{1}{h}\int_0^x f\left(\frac{s}{h}\right) ds = \int_0^{\frac{x}{h}} f(s)\, ds \leq \int_0^1 f(s)\, ds = C. \tag{2.62}
$$

For the remaining terms we have

$$
\begin{aligned}
&\frac{1}{h}\left[\int_0^{\kappa-x} f\left(\frac{\sqrt{x^2+s^2}}{h}\right) ds - \frac{1}{h}\int_0^{\kappa} f\left(\frac{s}{h}\right) ds\right] \\
=&\frac{1}{h}\left[\int_0^{\kappa} f\left(\frac{\sqrt{x^2+s^2}}{h}\right) ds - \frac{1}{h}\int_0^{\kappa} f\left(\frac{s}{h}\right) ds\right] - \underbrace{\frac{1}{h}\int_{\kappa-x}^{\kappa} f\left(\frac{\sqrt{x^2+s^2}}{h}\right) ds}_{D}.
\end{aligned}
$$

We can estimate D in the following way, where we use that f is positive and decreasing and that $x \leq h$:

$$\begin{aligned} |D| = \frac{1}{h}\int_{\kappa-x}^{\kappa} f\left(\frac{\sqrt{x^2+s^2}}{h}\right) ds \leq \frac{1}{h}\int_{\kappa-x}^{\kappa} f\left(\frac{s}{h}\right) ds \leq \frac{1}{h}\int_{\kappa-h}^{\kappa} f\left(\frac{s}{h}\right) ds \\ = \frac{1}{h}\cdot h \cdot f\left(\frac{\kappa-x}{h}\right) \leq f\left(\frac{\kappa-h}{h}\right) = f\left(\frac{\kappa}{h}-1\right) \to 0, \end{aligned} \tag{2.63}$$

and this is clearly uniform in x. It remains to estimate the remaining terms. We have

$$\begin{aligned} 0 \geq \frac{1}{h}\int_0^{\kappa}\left[f\left(\frac{\sqrt{x^2+s^2}}{h}\right) - f\left(\frac{s}{h}\right)\right] ds &= \int_0^{\frac{\kappa}{h}}\left[f\left(\sqrt{t^2+\left(\frac{x}{h}\right)^2}\right) - f(t)\right] dt \\ &\geq \int_0^{\frac{\kappa}{h}} f\left(\sqrt{1+t^2}\right) - f(t)\, dt \\ &\geq \int_0^{\infty} f\left(\sqrt{1+t^2}\right) - f(t)\, dt = -\frac{\pi}{4}, \end{aligned}$$

from which we can conclude that

$$\left|\frac{1}{h}\int_0^{\kappa}\left[f\left(\frac{\sqrt{x^2+s^2}}{h}\right) - f\left(\frac{s}{h}\right)\right] ds\right| \leq C, \tag{2.64}$$

and from this and the estimate on D we obtain:

$$\left|\frac{1}{h}\int_0^{\kappa-x} f\left(\frac{\sqrt{x^2+s^2}}{h}\right) ds + \frac{1}{h}\int_0^{x} f\left(\frac{s}{h}\right) ds - \frac{1}{h}\int_0^{\kappa} f\left(\frac{s}{h}\right) ds\right| \leq C, \tag{2.65}$$

in the case in which $x \leq h$.

Putting together the cases $x \leq h$ and $x \geq h$ we conclude the proof. □

Estimate for $\mathcal{G}_2$

We have the following result:

Lemma 2.10. *We have the following estimate for $\mathcal{G}_2$:*

$$|\mathcal{G}_2| \leq \frac{Ch}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right). \tag{2.66}$$

Proof. We have

$$\begin{aligned}|\mathcal{G}_2| &\leq \frac{h}{\eta^2|\log\varepsilon|}\int_{\partial\Omega}\int_{\partial\Omega}|\left(\overline{m}_h\cdot\nu\right)(x)|\frac{|\left(\overline{m}_h\cdot\nu\right)(x)-\left(\overline{m}_h\cdot\nu\right)(y)|}{|x-y|}dxdy\\ &\leq \frac{h}{\eta^2|\log\varepsilon|}\sum_{i,j=1}^{4}\int_{S_i}\int_{S_j}|\left(\overline{m}_h\cdot\nu\right)(x)|\frac{|\left(\overline{m}_h\cdot\nu\right)(x)-\left(\overline{m}_h\cdot\nu\right)(y)|}{|x-y|}dxdy,\end{aligned} \tag{2.67}$$

where $S_i, i = 1.\dots,4$ are the sides of the rectangle. We distinguish three cases:

- $i-j \not\equiv 1 \mod 4$ or $i-j\not\equiv 3 \mod 4$, i.e. the sides are not adjacent: then we use $|x-y|\geq \min\{a,b\}$ (where a and b denotes the sides' lengths) and the fact that $\|\overline{m}_h\|\leq 1$ to estimate:

$$\begin{aligned}&\frac{h}{\eta^2|\log\varepsilon|}\int_{S_i}\int_{S_j}|\left(\overline{m}_h\cdot\nu\right)(x)|\frac{|\left(\overline{m}_h\cdot\nu\right)(x)-\left(\overline{m}_h\cdot\nu\right)(y)|}{|x-y|}dxdy\\ &\qquad\leq \frac{h}{\eta^2|\log\varepsilon|}\frac{\max\{a^2,b^2\}}{\min\{a,b\}}\leq \frac{Ch}{\eta^2|\log\varepsilon|}\overline{E}_h\left(\overline{\mathbf{m}}_h\right),\end{aligned}$$

  where we have used the fact that $\liminf_{h\to 0}\overline{E}_h\left(\mathbf{m}_h\right)\geq 2\pi$, which we show below in Lemma 2.11.

- $i=j$, i.e. both sides are the same. Then the normal component is equal to either $\pm\overline{m}_{h,1}$ or $\pm\overline{m}_{h,2}$, according to the side, and each of these functions is in $H^1(\Omega)$. Since the domain Ω is Lipschitz we have that their traces are in $H^{1/2}(\partial\Omega)$ and that the trace operator is a bounded operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ (see [17]). For the corresponding seminorms we have the following inequality, for a constant $C > 0$:

$$\|\overline{m}_h\|_{\dot{H}^{1/2}(\partial\Omega)} \leq C\|\nabla\overline{m}_h\|_{L^2(\Omega)}.$$

Then can be seen easily as follows: let m be the average of $\overline{m}_h$ on Ω and notice that $\|\overline{m}_h - m\|_{\dot{H}^{1/2}(\partial\Omega)} = \|\overline{m}_h\|_{\dot{H}^{1/2}(\partial\Omega)}$. Now we can use the continuity of the trace operator and the Poincaré-Wirtinger inequality to obtain the conclusion.

Therefore we can estimate as follows:

$$\begin{aligned}
&\frac{h}{\eta^2||\log\varepsilon||}\int_{S_i}\int_{S_i}|(\overline{m}_h\cdot\nu)(x)|\frac{|(\overline{m}_h\cdot\nu)(x)-(\overline{m}_h\cdot\nu)(y)|}{|x-y|}dxdy\\
&\quad\leq\frac{Ch}{\eta^2|\log\varepsilon|}\|m_h\cdot\nu_i\|_{L^2(S_i)}\|m_{h,k}\|_{\dot{H}^{1/2}(S_i)}|S_i|^{1/2}\\
&\quad\leq\frac{C}{\eta^2|\log\varepsilon|}\|m_h\cdot\nu_i\|_{L^2(S_i)}\|m_{h,k}\|_{\dot{H}^{1/2}(\partial\Omega)}\\
&\quad\leq\frac{Ch}{\eta^2|\log\varepsilon|}\|m_h\cdot\nu_i\|_{L^2(S_i)}\|m_{h,k}\|_{\dot{H}^1(\Omega)}\\
&\quad\leq\frac{Ch}{\eta^2|\log\varepsilon|}\|m_h\cdot\nu_i\|_{L^2(S_i)}\|m_h\|_{\dot{H}^1(\Omega)}\\
&\quad\leq\frac{Ch}{\eta^2|\log\varepsilon|}\sqrt{\varepsilon|\log\varepsilon|\overline{E}_h(\overline{m}_h)}\sqrt{|\log\varepsilon|\overline{E}_h(m_h)}.
\end{aligned}\tag{2.68}$$

We now can write

$$\begin{aligned}\sum_{i=1}^{4}\frac{h}{\eta^2|\log\varepsilon|}\int_{S_i}\int_{S_i}|(\overline{m}_h\cdot\nu)(x)|\frac{|(\overline{m}_h\cdot\nu)(x)-(\overline{m}_h\cdot\nu)(y)|}{|x-y|}dxdy\\ \leq\sum_{i=1}^{4}\frac{Ch}{\eta^2|\log\varepsilon|}\|m_h\cdot\nu_i\|_{L^2(S_i)}\|m_h\|_{H^1(\Omega)}\\ \leq\frac{4Ch}{\eta^2|\log\varepsilon|}\sqrt{\varepsilon|\log\varepsilon|\overline{E}_h(\overline{m}_h)}\sqrt{|\log\varepsilon|\overline{E}_h(m_h)}\\ \leq\frac{4Ch\sqrt{\varepsilon}}{\eta^2}\overline{E}_h(\overline{m}_h)\,.\end{aligned}\tag{2.69}$$

- $i-j\equiv 1\mod 4$: in this case we use again that $\|\overline{m}_h\|\leq 1$ and estimate the integral as follows:

$$\begin{aligned}\frac{h}{\eta^2|\log\varepsilon|}\int_{S_i}\int_{S_j}|(\overline{m}_h\cdot\nu)(x)|\frac{|(\overline{m}_h\cdot\nu)(x)-(\overline{m}_h\cdot\nu)(y)|}{|x-y|}dxdy\\ \leq\frac{2h}{\eta^2|\log\varepsilon|}\int_0^a\int_0^b\frac{1}{\sqrt{s^2+t^2}}dsdt\leq\frac{C(a,b)\,h}{\eta^2|\log\varepsilon|}.\end{aligned}\tag{2.70}$$

To conclude the proof we need to show that $\overline{E}_h(\overline{\mathbf{m}}_h)\geq C>0$, which we have used above. This follows from Lemma 2.11 below. □

Lemma 2.11. *Let $\mathbf{m}_h\in H^1(\Omega;\mathbb{R}^3)$ be a sequence such that $|\mathbf{m}_h|\leq 1$. Then*

$$\liminf_{h\to 0}\overline{E}_h(\mathbf{m}_h)\geq 2\pi.\tag{2.71}$$

Proof. Without loss of generality we can assume that $\mathbf{m}_h=(m_h,0)$ (because this does not increase the energy) and that $\limsup_{h\to 0}\overline{E}_h(\mathbf{m}_h)\leq C$ (because this does not affect the lower bound). We then observe that since $|m_h|\leq 1$ we have:

$$\overline{E}_h(\mathbf{m}_h)\geq\frac{1}{|\log\varepsilon|}E_{\varepsilon,\eta}(m_h)\,,\tag{2.72}$$

that we have equality for unit-length vectors. Then we can apply the results in Chapter 3 to show that there exists $M_h \in \mathbb{S}^1$ such that

$$\frac{1}{|\log \varepsilon|} E_{\varepsilon,\eta}\left(m_h\right) \geq \frac{1}{|\log \varepsilon|} E_{\varepsilon,\eta}\left(M_h\right) - o\left(1\right). \tag{2.73}$$

Let us show this: we have that $E_{\varepsilon,\eta}\left(m_h\right) \leq C|\log \varepsilon|$ for $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$ (since $\limsup_{h\to 0} \overline{E}_h\left(\mathbf{m}_h\right) \leq C$), and so thanks to Theorem 3.7 we can find $M_h \in \mathbb{S}^1$ such that

$$E_{\varepsilon,\eta}\left(M_h\right) \leq E_{\varepsilon,\eta}\left(m_h\right) + \hat{C}\eta^{\tilde{\beta}}\left(E_{\varepsilon,\eta}\left(m_h\right) + \sqrt{E_{\varepsilon,\eta}\left(m_h\right)}\right).$$

Now we can divide by $|\log \varepsilon|$ both sides and get

$$\frac{E_{\varepsilon,\eta}\left(M_h\right)}{|\log \varepsilon|} \leq \frac{E_{\varepsilon,\eta}\left(m_h\right)}{|\log \varepsilon|} + C\eta^{\tilde{\beta}}\left(\frac{E_{\varepsilon,\eta}\left(m_h\right)}{|\log \varepsilon|} + \frac{1}{\sqrt{|\log \varepsilon|}}\sqrt{\frac{E_{\varepsilon,\eta}\left(m_h\right)}{|\log \varepsilon|}}\right). \tag{2.74}$$

Now using that $E_{\varepsilon,\eta}\left(m_h\right) \leq C|\log \varepsilon|$ we get

$$C\eta^{\tilde{\beta}}\left(\frac{E_{\varepsilon,\eta}\left(m_h\right)}{|\log \varepsilon|} + \frac{1}{\sqrt{|\log \varepsilon|}}\sqrt{\frac{E_{\varepsilon,\eta}\left(m_h\right)}{|\log \varepsilon|}}\right) = o\left(1\right).$$

From this and (2.74) we then get (2.73). Now if $\varphi_\varepsilon \in H^1\left(\Omega\right)$ is a lift of M_h, i.e. such that $M_h = e^{i\varphi_\varepsilon}$ (see Bethuel-Zheng [8, Lemma 4] for the existence of such a lift) we obtain that

$$\frac{1}{|\log \varepsilon|} E_{\varepsilon,\eta}\left(M_h\right) - o\left(1\right) = \frac{1}{|\log \varepsilon|} E_{\varepsilon}\left(\varphi_\varepsilon\right) - o\left(1\right). \tag{2.75}$$

Now we estimate the energy from below by the energy of a minimizer $\hat{u}_\varepsilon$ and the use the lower bound for the energy in Theorem 5.16:

$$\begin{aligned} \liminf_{h\to 0} \overline{E}_h\left(m_h\right) &\geq \liminf_{\varepsilon\to 0}\left(\frac{1}{|\log \varepsilon|} E_{\varepsilon}\left(\varphi_\varepsilon\right) - o\left(1\right)\right) \\ &\geq \liminf_{\varepsilon\to 0}\left(\frac{E_{\varepsilon}\left(\hat{u}_\varepsilon\right)}{|\log \varepsilon|} - o\left(1\right)\right) = 2\pi. \end{aligned} \tag{2.76}$$

□

We can then prove Lemma 2.6:

Proof. (of Lemma 2.6) From Lemma 2.7 we get that:

$$\left|\frac{\mathcal{G}_1}{4\pi}-\frac{1}{2\pi\varepsilon|\log\varepsilon|}\int_{\partial\Omega}\left(\overline{m}_h\cdot\nu\right)^2 d\mathcal{H}^1\right| \ll \frac{Ch}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right).$$

From Lemma 2.10 we have:

$$|\mathcal{G}_2| \leq \frac{Ch}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right).$$

Then putting these two together we have using (2.35) that:

$$\left|\frac{\mathcal{C}_2}{4\pi\eta^2 h|\log\varepsilon|}-\frac{1}{2\pi\varepsilon|\log\varepsilon|}\int_{\partial\Omega}\left(\overline{m}_h\cdot\nu\right)^2 d\mathcal{H}^1\right| \leq \frac{Ch}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right),$$

which concludes the proof. □

We can finally give the proof of Proposition 2.4 and hence of Theorem 2.1.

Proof of Proposition 2.4. We need to estimate the following quantity:

$$\frac{1}{|\log\varepsilon|}\left|\frac{1}{\eta^2 h}\int_{\mathbb{R}^3}|\nabla\overline{U}_h|^2 d\mathbf{x}-\frac{1}{\eta^2}\int_{\Omega}\overline{m}_{h,3}^2 dx-\frac{1}{2\pi\varepsilon}\int_{\partial\Omega}\left(\overline{m}_h\cdot\nu\right)^2 d\mathcal{H}^1\right|.$$

We can rewrite this using (2.24) as

$$\frac{1}{|\log\varepsilon|}\left|\frac{1}{4\pi\eta^2 h}\left(\mathcal{A}+2\mathcal{B}+\mathcal{C}\right)-\frac{1}{\eta^2}\int_{\Omega}\overline{m}_{h,3}^2 dx-\frac{1}{2\pi\varepsilon}\int_{\partial\Omega}\left(\overline{m}_h\cdot\nu\right)^2 d\mathcal{H}^1\right|. \tag{2.77}$$

We now have the following estimates, which we obtain from (2.28) and (2.29):

$$\frac{|\mathcal{A}|}{4\pi|\log\varepsilon|\eta^2 h} \leq \frac{Ch^2|\log\varepsilon|\overline{E}_h\left(\overline{\mathbf{m}}_h\right)}{4\pi|\log\varepsilon|\eta^2 h} \leq \frac{Ch}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right). \tag{2.78}$$

$$\frac{|\mathcal{B}|}{4\pi|\log\varepsilon|\eta^2 h} \leq \frac{Ch^2\varepsilon^{1/2}|\log\varepsilon|\overline{E}_h\left(\overline{\mathbf{m}}_h\right)}{4\pi|\log\varepsilon|\eta^2 h} \leq \frac{Ch\varepsilon^{1/2}}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right). \tag{2.79}$$

We now have to estimate the quantity

$$\frac{1}{|\log \varepsilon|}\left|\frac{1}{4\pi\eta^2 h}\mathcal{C} - \frac{1}{\eta^2}\int_\Omega \overline{m}_{h,3}^2 dx - \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1\right|. \tag{2.80}$$

We write $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined as in (2.30). Then we can use Lemma 2.5 to get

$$\frac{1}{\eta^2|\log \varepsilon|}\left|\frac{\mathcal{C}_1}{4\pi h} - \int_\Omega \overline{m}_{h,3}^2 (x)\, dx\right| \leq C\frac{h}{\eta^2}\left(\frac{\log \frac{\eta^2}{h}}{|\log \varepsilon|} + 1\right) \overline{E}_h\left(\overline{\mathbf{m}}_h\right), \tag{2.81}$$

and Lemma 2.6 to get

$$\frac{1}{|\log \varepsilon|}\left|\frac{\mathcal{C}_2}{4\pi\eta^2 h} - \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1\right| \leq C\frac{h}{\eta^2}\overline{E}_h\left(\overline{\mathbf{m}}_h\right). \tag{2.82}$$

Now if we combine (2.81) and (2.82) we get the following estimate for the quantity in (2.80)

$$\frac{1}{|\log \varepsilon|}\left|\frac{1}{4\pi\eta^2 h}\mathcal{C} - \frac{1}{\eta^2}\int_\Omega \overline{m}_{h,3}^2 dx - \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \left(\overline{m}_h \cdot \nu\right)^2 d\mathcal{H}^1\right| \leq C\frac{h}{\eta^2}\left(\frac{\log \frac{\eta^2}{h}}{|\log \varepsilon|} + 1\right) \overline{E}_h\left(\overline{\mathbf{m}}_h\right). \tag{2.83}$$

Now by combining the estimates in (2.78), (2.79) and (2.83) we get the conclusion. □

Proof of Theorem 2.1. We need to compare the two energies energies (1.7) and (2.3). Using (2.13) we we can compare the Dirichlet energies as follows:

$$\int_{\Omega_h} |\nabla \overline{\mathbf{m}}_h|^2 dx \leq \frac{1}{h}\int_{\boldsymbol{\Omega}_h} |\nabla \mathbf{m}_h|^2 d\mathbf{x}. \tag{2.84}$$

Then we can estimate the difference between the remaining terms of the energy as follows:

$$\left|\frac{1}{\eta^2 h|\log\varepsilon|}\int_{\mathbb{R}^3}|\nabla U_h|^2 d\mathbf{x} - \frac{1}{\eta^2|\log\varepsilon|}\int_\Omega\left(1-|\overline{m}_h|^2\right)dx - \frac{1}{2\pi\varepsilon|\log\varepsilon|}\int_{\partial\Omega}\left(\overline{m}_h\cdot\nu\right)^2 d\mathcal{H}^1\right|$$
$$\leq \frac{1}{\eta^2 h|\log\varepsilon|}\left|\int_{\mathbb{R}^3}|\nabla U_h|^2 d\mathbf{x} - \int_{\mathbb{R}^3}|\nabla\overline{U}_h|^2 d\mathbf{x}\right|$$
$$+\left|\frac{1}{\eta^2|\log\varepsilon|}\left(\int_\Omega \overline{m}_{h,3}^2 dx - \int_\Omega\left(1-|\overline{m}_h|^2\right)dx\right)\right|$$
$$+\frac{1}{|\log\varepsilon|}\left|\frac{1}{\eta^2 h}\int_{\mathbb{R}^3}|\nabla\overline{U}_h|^2 d\mathbf{x} - \frac{1}{\eta^2}\int_\Omega \overline{m}_{h,3}^2 dx - \frac{1}{2\pi\varepsilon}\int_{\partial\Omega}\left(\overline{m}_h\cdot\nu\right)^2 d\mathcal{H}^1\right| \tag{2.85}$$

Now the conclusion of Theorem 2.1 follows combining the estimate (2.84) and the results of Lemma 2.3, Lemma 2.2 and Proposition 2.4, which we use to estimate the right-hand side in (2.85) and get:

$$\left|\frac{1}{\eta^2 h|\log\varepsilon|}\int_{\mathbb{R}^3}|\nabla U_h|^2 d\mathbf{x} - \frac{1}{\eta^2|\log\varepsilon|}\int_\Omega\left(1-|\overline{m}_h|^2\right)dx - \frac{1}{2\pi\varepsilon}\int_{\partial\Omega}\left(\overline{m}_h\cdot\nu\right)^2 d\mathcal{H}^1\right|$$
$$\leq \frac{Ch}{\eta^2}\sqrt{\frac{E_h(\mathbf{m}_h)}{|\log\varepsilon|}} + \frac{Ch}{\eta^2}\sqrt{\frac{E_h(\mathbf{m}_h)}{|\log\varepsilon|}} + C\frac{h}{\eta^2}\left(1+\frac{\log\frac{\eta^2}{h}}{|\log\varepsilon|}\right)\overline{E}_h(\overline{\mathbf{m}}_h)$$
$$\leq\left(\overline{E}_h(\overline{\mathbf{m}_h}) + \sqrt{\frac{E_h(\mathbf{m}_h)}{|\log\varepsilon|}}\right)O(A(h)),$$

from which the conclusion of Theorem 2.1 follows. □

We conclude the chapter with the following result:

Theorem 2.12. *Under the same assumptions of Theorem 2.1 we have that*

$$E_h(\mathbf{m}_h) \geq \overline{E}_h(\overline{\mathbf{m}_h}) - o(1) \qquad \text{as } h\to 0. \tag{2.86}$$

In the more restrictive regime $\frac{\log|\log h|}{|\log h|} \ll \varepsilon$ *we have the improved estimate:*

$$E_h(\mathbf{m}_h) \geq \overline{E}_h(\overline{\mathbf{m}_h}) - o\left(\frac{1}{|\log\varepsilon|}\right) \qquad \text{as } h\to 0. \tag{2.87}$$

Proof. The proof is analogous to [25, Theorem 1]: since $\limsup_{h\to 0} E_h(\mathbf{m}_h) < \infty$ we can find $K > 0$ such that $E_h(\mathbf{m}_h) \leq K$ for all $h < h_0$ for some $h_0 > 0$. From Theorem 2.1 we get that

$$E_h(\mathbf{m}_h) \geq \overline{E}_h(\overline{\mathbf{m}_h}) - \left(\overline{E}_h(\overline{\mathbf{m}_h}) + \sqrt{\frac{K}{|\log \varepsilon|}}\right) O(A(h)), \tag{2.88}$$

where $A(h)$ is defined as in Theorem 2.1. We have that in our regime $A(h) = o(1)$ as $\varepsilon \to 0$, as was shown on page 22. We conclude that

$$\limsup_{h\to 0} \overline{E}_h(\overline{\mathbf{m}_h}) \leq K, \tag{2.89}$$

and from this we can easily conclude that $E_h(\mathbf{m}_h) \geq \overline{E}_h(\overline{\mathbf{m}_h}) - o(1)$, since we have the estimate

$$E_h(\mathbf{m}_h) \geq \overline{E}_h(\overline{\mathbf{m}_h}) - \left(K + \sqrt{\frac{K}{|\log \varepsilon|}}\right) O(A(h)). \tag{2.90}$$

In the more restrictive regime $\frac{\log|\log h|}{|\log h|} \ll \varepsilon$ we have from (2.12) that

$$A(h) = \frac{\log|\log h|}{\varepsilon|\log \varepsilon||\log h|} \ll \frac{1}{|\log \varepsilon|}, \tag{2.91}$$

and so we obtain that

$$E_h(\mathbf{m}_h) \geq \overline{E}_h(\overline{\mathbf{m}_h}) - o\left(\frac{1}{|\log \varepsilon|}\right) \qquad \text{as } h \to 0. \tag{2.92}$$

This concludes the proof.

□

Chapter 3

Reduction to $\mathbb{S}^1$-valued maps

In this chapter we employ the strategy devised by Ignat and Kurzke [26] to show that we can replace the (in-plane component of the) averaged magnetization $\overline{\mathbf{m}}_h$ with a unit length magnetization M_h which takes values in $\mathbb{S}^1$, without increasing the energy asymptotically. This is important to complete the reduction from the full micromagnetic energy to the scalar energy functionals

$$E_\varepsilon(u) := \int_\Omega |\nabla u|^2 dx + \frac{1}{2\varepsilon}\int_{\partial\Omega} \sin^2(u-g)\, d\mathcal{H}^1. \tag{3.1}$$

Indeed we prove that maps $m = m_\varepsilon : \Omega \to \mathbb{R}^2$ with energy of order $E_{\varepsilon,\eta}(m) \leq C|\log\varepsilon|$ – where $E_{\varepsilon,\eta}$ was defined in Chapter 1 – can be approximated by suitable $\mathbb{S}^1$-valued maps $M = M_\varepsilon : \Omega \to \mathbb{S}^1$ in the regime $|\log\varepsilon| \ll |\log\eta|$ (see in particular (37) in [26, Theorem 3.1], which we prove below in Theorem 3.7). The proof follows several steps: we give a brief overview, before delving into the details. The idea is to subdivide the domain into small cells, and define a new function on a subdomain of the rectangle, via solving a minimization problem for the Ginzburg-Landau energy on each cell and putting everything together. We will be able to construct from this a unit-length vector on the same set. Then we will extend the definition of this function on the whole rectangle and show that this is close in energy to the original function.

3.1 Construction of the grid

Assume without loss of generality that our domain is the rectangle $Q = (-a, a) \times (-b, b)$, for $a, b > 0$, and let $\mu = b/a$. As a first step, we construct a grid in the domain which shifts by $R, \mu R$ along the x and y axes respectively, for $R \in \left(0, \eta^{\beta}\right)$, for $\beta \in (0, 1)$. Our construction is similar to that of Ignat and Kurzke [26], who work with a disc. To construct a grid which is homothetic to the whole domain (this will be important later) we slightly modify their construction. For every $R \in \left(0, \eta^{\beta}\right)$ define a set which consists of lines parallel to the y axis (i.e. where the shift is in the x direction) as

$$V_R^1 := \{(\pm x, y) \in Q : x > 0 \text{ and } x \in \left(\eta^{\beta}, a\right), x \equiv R \mod \eta^{\beta}\}.$$

We can define analogously a net V_y as

$$V_R^2 := \{(x, \pm y) \in Q : |y| \in \left(\mu\eta^{\beta}, b\right), y \equiv \mu R \mod \mu\eta^{\beta}\}.$$

Let $\mathcal{R}_R$ the largest rectangle symmetrical with respect to the coordinate axes and whose boundary is contained in $V_R^1 \cup V_R^2$. Define $\mathcal{G}_R := \overline{\mathcal{R}_R} \cap (V_R^1 \cup V_R^2)$. Then $\mathcal{G}_R$ is a rectangular grid, and the area enclosed in it is a rectangle similar to Q, whose boundary is $\mathcal{R}_R$. Another equivalent way of defining $\mathcal{G}_R$ is to set $X_{max} := \max\{|x| : (x, y) \in V_R^1\}, Y_{max} := \max\{|y| : (x, y) \in V_R^2\}$ and define $\mathcal{G}_R := \{(x, y) \in V_R^1 \cup V_R^2 : |x| \leq X_{max}, |y| \leq Y_{max}\}$. By using Fubini's theorem we can estimate the integral of the bulk term as follows:

$$\int_Q e_\eta (m)\, dx \geq \int_0^{\eta^\beta} f(r)\, dr,$$

where $e_\eta (m)$ is the Ginzburg-Landau energy density

$$e_\eta (m) = |\nabla m|^2 + \frac{1}{\eta^2} \left(1 - |m|^2\right)^2, \tag{3.2}$$

and the function $f : \left(0, \eta^{\beta}\right) \to \mathbb{R}_+$ is defined as

$$f(r) := \int_{\mathcal{G}_r} e_\eta(m)\, d\mathcal{H}^1.$$

Now using the mean value theorem we establish the existence of a shift $R \in (0, \eta^\beta)$ such that

$$\int_0^{\eta^\beta} f(r) = \eta^\beta f(R).$$

So in the end we conclude that

$$\frac{1}{\eta^\beta}\int_Q e_\eta(m)\, dx \geq \int_{\mathcal{G}_R} e_\eta(m)\, d\mathcal{H}^1.$$

So we have obtained a rectangular grid $\mathcal{G}_R$ such that

$$\int_{\mathcal{G}_R} e_\eta(m)\, d\mathcal{H}^1 \leq \frac{1}{\eta^\beta}\int_Q e_\eta(m)\, dx \lesssim \frac{E_{\varepsilon,\eta}(m)}{\eta^\beta}. \tag{3.3}$$

We observe that all the cells have size $\sim \eta^{2\beta}$. For any cell $\mathcal{C} \subset \mathcal{G}_R$ we define as $\operatorname{int}(\mathcal{C})$ the 2-dimensional region bounded by $\mathcal{C}$. If we consider the union of all the cells we obtain a rectangular region $\mathcal{G}_R$ such that $\operatorname{int}(\mathcal{G}_R)$ is contained in $\operatorname{int}(Q)$ at a distance less than $C\eta^\beta$ from the boundary ∂Q, for a constant C which only depends on μ.

3.2 Construction of an $\mathbb{S}^1$-valued function

Now that we have constructed the grid we can define the required $\mathbb{S}^1$-valued replacement. In the interior of any cell $\mathcal{C}$ we define a new function that coincides with the original one on the boundary of the cell and that minimizes the Ginzburg-Landau energy. This means that we define a function $w = w_\varepsilon \in H^1(\operatorname{int}(\mathcal{C}), \mathbb{R}^2)$ to be a solution of

$$\min_{w=m \text{ on } \mathcal{C}} \int_{\operatorname{int}(\mathcal{C})} e_\eta(w)\, dx. \tag{3.4}$$

This defines w on the whole of $\mathrm{int}\,(\mathcal{G}_R)$. The next important step is to show that as ε tends to 0 the absolute value of function w_ε approaches 1, uniformly. We do this using a result employed also in [26] (see Proposition 3.2), which was obtained by Ignat, Kurzke and Lamy [27, Corollary 4]:

Proposition 3.1 ([26, 27])**.** *For a sequence/family $\eta \to 0$ let $\Omega := \left(0, \eta^\beta\right) \times \left(0, \eta^\beta\right) \subset \mathbb{R}^2$ with $\beta \in (0,1)\,, g_\eta \in H^1(\partial\Omega)$ and let $w_\eta \in H^1(\Omega, \mathbb{R}^2)$ be a minimizer of*

$$\min_{w = g_\eta \text{ on } \partial\Omega} \int_\Omega e_\eta(w)\,, \tag{3.5}$$

where e_η is the Ginzburg-Landau energy density defined in (3.2). Let $\kappa = \kappa(\eta) \ll |\log \eta|$ as $\eta \to 0$. Assume that

$$\int_{\partial\Omega} |\partial_\tau g_\eta|^2 + \frac{1}{\eta^2}\left(1 - |g_\eta|^2\right)^2 d\mathcal{H}^1 \leq \frac{\kappa}{\eta^\beta} \quad \text{and} \quad \int_\Omega e_\eta(w_\eta)\,dx \leq \kappa. \tag{3.6}$$

Then there exists $0 < \tilde{\beta} < \frac{1-\beta}{6}$ such that for the terms w_η in the sequence with $\eta \leq \eta_0$ (η_0 depends only on Ω):

$$\sup_\Omega \left||w_\eta|^2 - 1\right| \leq C\eta^{\tilde{\beta}}, \tag{3.7}$$

where $C > 0$ depends only on Ω. In particular w_η has degree 0 on $\partial\Omega$.

Applying this result to $w = w_\eta$ defined in (3.4) with $\kappa = |\log \varepsilon| \ll |\log \eta|$ (from $E_{\varepsilon,\eta}(m) \leq C|\log \varepsilon|$ and (3.3) we have that (3.6) holds) we obtain that there exists $\tilde{\beta} \in \left(0, \frac{1-\beta}{6}\right)$ such that for some $\tilde{C} > 0$:

$$\sup_{\mathrm{int}(\mathcal{G}_R)} \left\||w|^2 - 1\right\| \leq \tilde{C}\eta^{\tilde{\beta}} =: \delta \ll 1. \tag{3.8}$$

This in particular means that for small ε (and therefore for small η) we have $|w| \geq \frac{1}{2}$ on $\mathcal{G}_R$ and $\deg(w, \mathcal{C}) = 0$ on each cell $\mathcal{C}$. Thus we can define a unit length function as:

$$\hat{M} = \frac{w}{|w|} \text{ in int} \left(\mathcal{G}_R\right). \tag{3.9}$$

Notice that $|w|^2|\nabla\hat{M}|^2 \leq |\nabla w|^2$ and so, for small $\varepsilon > 0$, we deduce that:

$$\begin{aligned} \int_{\text{int}(\mathcal{G}_R)} |\nabla\hat{M}|^2 dx &\leq (1+2\delta) \int_{\text{int}(\mathcal{G}_R)} |\nabla w|^2 dx \\ &\leq (1+2\delta) \int_{\text{int}(\mathcal{G}_R)} e_\eta\left(w\right) dx \\ &\overset{(\dagger)}{\leq} (1+2\delta) \int_{\text{int}(\mathcal{G}_R)} e_\eta\left(m\right) dx \\ &\leq (1+2\delta) \int_Q e_\eta\left(m\right) dx, \end{aligned} \tag{3.10}$$

where in ($\dagger$) we used the fact that w is a minimizer of the energy by (3.4).

3.3 Construction of the function on all of Q

We have defined the function on $\text{int}\left(\mathcal{G}_R\right)$: the next step is to extend it to the whole of Q. This is easily done by noticing that the two domains are similar, i.e. related by a homothetic transformation. Indeed we have that $Q = \left(1 + O\left(\eta^\beta\right)\right) \text{int}\left(\mathcal{G}_R\right)$. We can then define a function M on Q as follows [1]:

$$M\left(x\right) = \hat{M}\left(\hat{x}\right), \tag{3.11}$$

where $x = \left(1+\eta^\beta\right)\hat{x}$ for all $\hat{x} \in \text{int}\left(\mathcal{G}_R\right)$.

Our goal is then to show that the energy does not change asymptotically if we replace m with the new function M which we have constructed. The first step is to show an estimate for the Dirichlet energy. We have the following lemma

[1] we assume for simplicity that $Q = \left(1+\eta^\beta\right)\text{int}\left(\mathcal{G}_R\right)$: in the general case the argument is carried out in the same way, but the notation becomes more cumbersome.

Lemma 3.2. *For the function M defined as above we have the following estimate:*

$$\int_Q |\nabla M|^2 dx \leq (1+2\delta) \int_Q e_\eta(m)\, dx. \tag{3.12}$$

Proof. The proof is a simple calculation, namely

$$\int_Q |\nabla M|^2 dx \leq \int_{\mathrm{int}(\mathcal{G}_R)} |\nabla \hat{M}|^2 \overset{(\dagger)}{\leq} (1+2\delta) \int_Q e_\eta(m)\, dx, \tag{3.13}$$

where the inequality (†) follows from (3.10). □

The next important step is to estimate the L^2 distance between m and M, which will be relevant to estimating the second term in the reduced energy (2.3). We start first by proving the estimate for $m - \hat{M}$ in $\mathrm{int}(\mathcal{G}_R)$. We have the following result:

Lemma 3.3. *For $\hat{M}$ and m in $int(\mathcal{G}_R)$ we have that*

$$\int_{int(\mathcal{G}_R)} |\hat{M} - m|^2 dx \lesssim \eta^{2\beta} \int_Q e_\eta(m)\, dx. \tag{3.14}$$

Proof. By the Poincaré-Wirtinger inequality we have that on each cell $\mathcal{C} \subset \mathcal{G}_R$

$$\int_{\mathrm{int}(\mathcal{C})} \left| \hat{M} - \fint_{\mathcal{C}} \hat{M} \right|^2 dx \lesssim \eta^{2\beta} \int_{\mathrm{int}(\mathcal{C})} |\nabla \hat{M}|^2 dx \tag{3.15}$$

and

$$\int_{\mathrm{int}(\mathcal{C})} \left| m - \fint_{\mathcal{C}} m \right|^2 dx \lesssim \eta^{2\beta} \int_{\mathrm{int}(\mathcal{C})} |\nabla m|^2 dx \tag{3.16}$$

where $\fint_{\mathcal{C}}$ denotes the average over the cell $\mathcal{C}$. We recall that $|m| \geq \frac{1}{2}$ on $\mathcal{G}_R$ we can set $v := \frac{m}{|m|}$ on $\mathcal{G}_R$ and we have clearly that $|v| = 1$. Therefore we have that $v = \hat{M}$ on $\mathcal{G}_R$ and Jensen's inequality yields

$$\begin{aligned}\int_{\mathrm{int}(\mathcal{C})}\left|\fint_{\mathcal{C}}\left(\hat{M}-m\right)d\mathcal{H}^1\right|^2dx &= \int_{\mathrm{int}(\mathcal{C})}\left|\fint_{\mathcal{C}}\left(v-v|m|\right)d\mathcal{H}^1\right|^2dx\\ &\lesssim \eta^{2\beta}\fint_{\mathcal{C}}\left(1-|m|\right)^2d\mathcal{H}^1\\ &\lesssim \eta^{\beta}\int_{\mathcal{C}}\left(1-|m|^2\right)^2d\mathcal{H}^1\\ &\lesssim \eta^{\beta+2}\int_{\mathcal{C}}e_\eta\left(m\right)d\mathcal{H}^1.\end{aligned}\tag{3.17}$$

Adding these inequalities and using (3.3) and (3.10) we obtain the conclusion (observe that $\eta^{2+2\beta}\ll\eta^{2\beta}$, since $\eta\ll 1$). □

We now use the result of Lemma 3.3 to obtain an L^2-estimate on $M-m$ on Q. We summarize this in the following lemma:

Lemma 3.4. *With M and m as above we have that*

$$\int_Q|M-m|^2dx\lesssim\eta^{2\beta}\int_Q e_\eta\left(m\right)dx.\tag{3.18}$$

Proof. From Lemma 3.3 we have that

$$\int_Q|M-\tilde{m}|^2dx=\int_{\mathrm{int}(\mathcal{G}_R)}|\hat{M}-m|^2dx=\lesssim\eta^{2\beta}\int_Q e_\eta\left(m\right)dx,\tag{3.19}$$

where $\tilde{m}$ is defined on Q as $\hat{m}\left(x\right)=m\left(\hat{x}\right)$, where $\hat{x}=\frac{1}{1+\eta^\beta}x$ for all $x\in Q$. So it is enough to show that the same estimate holds for $m-\hat{m}$ as well. This is done by a easy calculation that can be found in [26, Theorem 3.1, Step 6]; we will not repeat it here.

□

3.4 Bounds on the boundary terms

To complete the estimate of the energy we need to obtain bounds for the boundary term. We will show that the following holds:

Lemma 3.5. *The boundary term satisfies*

$$\frac{1}{2\pi\varepsilon}\int_{\partial Q}(M\cdot\nu)^2\,d\mathcal{H}^1\le\frac{1}{2\pi\varepsilon}\int_{\partial Q}(m\cdot\nu)^2\,d\mathcal{H}^1+\frac{c\eta^{\beta/2}}{\varepsilon}\sqrt{\int_Q e_\eta(m)}. \tag{3.20}$$

Proof. We have that (using that $a^2\le b^2+2|a-b|,\forall a\in[-1,1]\,,b\in\mathbb{R}$ – see [26] on page 26)

$$\begin{aligned}\frac{1}{2\pi\varepsilon}\int_{\partial Q}(M\cdot\nu)^2\,d\mathcal{H}^1&\le\frac{1}{2\pi\varepsilon}\int_{\partial Q}(m\cdot\nu)^2\,d\mathcal{H}^1+\frac{1}{\pi\varepsilon}\int_{\partial Q}|(M-m)\cdot\nu|d\mathcal{H}^1\\&\le\frac{1}{2\pi\varepsilon}\int_{\partial Q}(m\cdot\nu)^2\,d\mathcal{H}^1+\frac{c}{\varepsilon}\|M-m\|_{L^2(\partial Q)}.\end{aligned} \tag{3.21}$$

Hence what we need to complete the proof is to show the appropriate bound for the right hand side. The conclusion then follows from the following Lemma 3.6. □

In the previous Lemma 3.5 we used the following result:

Lemma 3.6. *On the boundary ∂Q we have the following L^2-estimate for $M-m$:*

$$\int_{\partial Q}|M-m|^2 d\mathcal{H}^1\lesssim\eta^\beta\int_Q e_\eta(m)\,dx. \tag{3.22}$$

Proof. Let $\lambda\in(0,1)$ be the largest number such that $\partial Q_\lambda\subset\mathcal{G}_R$, where $Q_\lambda:=\lambda Q$. By our assumption we have chosen $\lambda=\frac{1}{1+\eta^\beta}$. We have that

$$\int_{\partial Q}|M(x)-m(x)|^2 d\mathcal{H}^1(x)=\left(1+\eta^\beta\right)\int_{\partial Q_\lambda}\left|m\left(\frac{\hat{x}}{\lambda}\right)-\hat{M}(\hat{x})\right|^2 d\mathcal{H}^1(\hat{x}). \tag{3.23}$$

To obtain an estimate on this integral, we estimate separately the two quantities

$$\int_{\partial Q_\lambda}|\hat{M}-m|^2 d\mathcal{H}^1\quad\text{and}\quad\int_{\partial Q_\lambda}\left|m\left(\frac{x}{\lambda}\right)-m(x)\right|^2 d\mathcal{H}^1(x).$$

For the first term, since $\hat{M} = v$ on $\mathcal{G}_R$ and $|v| = 1$ and $m = |m|v$ on $\mathcal{G}_R$, we have:

$$\begin{aligned}\int_{\partial Q_\lambda} |\hat{M} - m| d\mathcal{H}^1 &= \int_{\partial Q_\lambda} (1 - |m|)^2 \, d\mathcal{H}^1 \\ &\leq \eta^2 \int_{\mathcal{G}_R} e_\eta (m) \lesssim \eta^{2-\beta} \int_Q e_\eta (m) \, .\end{aligned} \tag{3.24}$$

As regards the second term we have

$$\begin{aligned}\int_{\partial Q_\lambda} \left| m \left(\frac{x}{\lambda} \right) - m(x) \right|^2 d\mathcal{H}^1 (x) &= \lambda \int_{\partial Q} |m(\lambda y) - m(y)|^2 d(y) \, d\mathcal{H}^1 (y) \\ &= \lambda \int_{\partial Q} \left(\int_\lambda^1 y \cdot \nabla m(ty) \, dt \right)^2 d\mathcal{H}^1 .\end{aligned} \tag{3.25}$$

By Jensen's inequality this can be estimated as follows:

$$\begin{aligned}&\lambda \int_{\partial Q} \left(\int_\lambda^1 y \cdot \nabla m(ty) \, dt \right)^2 d\mathcal{H}^1 \\ &\quad = \lambda \int_{\partial Q} (1 - \lambda)^2 \left(\frac{1}{1 - \lambda} \int_\lambda^1 y \cdot \nabla m(ty) \, dt \right)^2 d\mathcal{H}^1 \\ &\quad \leq \lambda (1 - \lambda) \int_{\partial Q} \int_\lambda^1 |y \cdot \nabla m(ty)|^2 dy \\ &\quad \lesssim \eta^\beta \int_{Q \setminus Q_\lambda} |\nabla m|^2 dx \lesssim \eta^\beta \int_Q e_\eta (m) \, dx.\end{aligned} \tag{3.26}$$

Since $\beta \in (0, 1)$, and so $2 - \beta > \beta$, and $\eta \ll 1$ we can conclude $\eta^{2-\beta} \ll \eta^\beta$, and we get the conclusion combining the estimates in (3.24) and (3.26).

□

We can now prove the main result of this chapter, namely that by replacing m with the unit-length version M we do not increase the energy in the limit. We state this as:

Theorem 3.7. *The energies of m and M satisfy, for a constant*

$$E_{\varepsilon,\eta}\left(M\right) \leq E_{\varepsilon,\eta}\left(m\right) + \hat{C}\eta^{\tilde{\beta}}\left(E_{\varepsilon,\eta}\left(m\right) + \sqrt{E_{\varepsilon,\eta}\left(m\right)}\right). \tag{3.27}$$

Proof. The conclusion follows by combining the results of Lemma 3.10, Lemma 3.4 and and Lemma 3.5. When applying Lemma 3.5 we observe that since $|\log\varepsilon| \ll |\log\eta|$ and $\beta > 1/2$ we can choose $\varepsilon_0 > 0$ such that $\frac{\eta^{\beta/2}}{\varepsilon} \leq \delta$ for every $\varepsilon < \varepsilon_0$. This concludes the proof. □

Chapter 4

Solutions in a corner

In the next chapters we want to prove some Γ-convergence results for minimizers in a rectangle. This should help us prove more rigorously some results on C and S states whose proof was sketched in the final chapter of [34]. These are micromagnetic states known from numerical experiments (see for example Rave and Hubert [24]). They correspond to local minimizers of the energy whose boundary conditions tend in the limit to the following configuration respectively[1]:

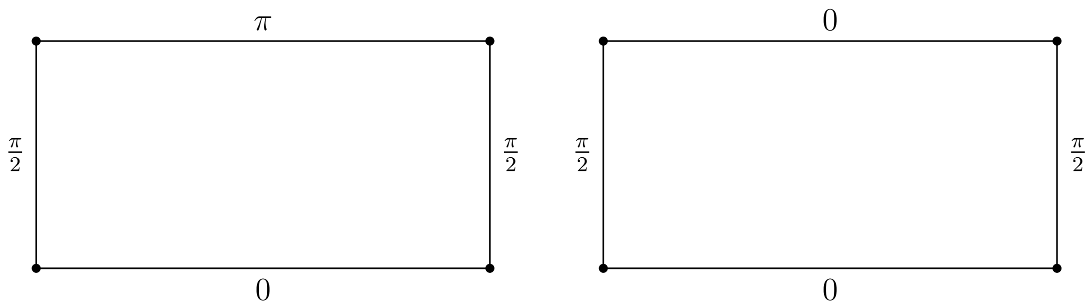


Figure 4.1: Limit boundary conditions for C and S states

These are values for the phase u that correspond to a magnetization $m = e^{iu}$ vector which is tangent along the sides (we observe that these configurations will

[1] We have two different kind of C state, which depend on whether the two sides where we have the same boundary values are the shorter or the longer ones.

have infinite energy, since the magnetization, or equivalently the phase, has a jump in the corners).

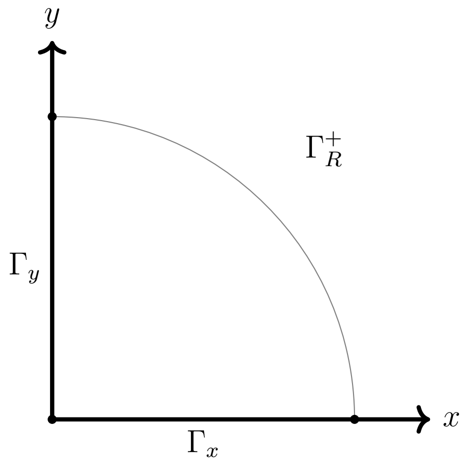
The energy functionals that we will study in this section have the form

$$\int_{\Omega} |\nabla u|^2 dx + \sum_{k=1}^{4} \frac{1}{2\pi\varepsilon} \int_{L_k} \sin^2(u - \alpha_k)\, d\mathcal{H}^1, \tag{4.1}$$

where $L_k, k = 1, \ldots, 4$ denote the sides of the rectangle, $\alpha_i \in \{m\frac{\pi}{2} : m \in \mathbb{Z}\}$ and where $|\alpha_k - \alpha_j| = \frac{\pi}{2}$ if $k - j \equiv 1 \mod 4$. If τ_k is the tangent vector to L_k then $e^{i\alpha_k} = \tau_k$.

We start with an observation: given a function $u \in H^1(\Omega)$ which is bounded, we can obtain a new function $u^* \in H^1(\Omega)$ and such that $0 \leq u^* \leq \pi$ in a way that does not increase the energy. This is done by 'reflecting' the function across its level sets $u = k\pi$: this does not increase the energy (since this reflection is analogous to taking the absolute value, and this does not change the value of the penalty term - since it is an even function) and does not increase the value of the Dirichlet energy, since the weak gradient satisfies $\nabla|u| = (\operatorname{sgn}(u)\nabla u)\,\chi_{\{u\neq 0\}}$. By repeating this a finite number of times (which we can do because u is bounded) we obtain the desired conclusion.

In particular this tells us that for any minimizer we can find another minimizer which lies between 0 and π. This of course does not imply that any minimizer must satisfy this property, since we have no uniqueness result for minimizers. In Chapter 7 we will see that near a corner we can in the same way assume that any minimizer lies between 0 and $\frac{\pi}{2}$. This justifies the assumptions that we will make in this chapter.



4.1 Uniqueness of minimizer in a right angle

In this section we consider an explicit solution of the blown-up Euler-Lagrange equation near a corner, and show that it is the unique solution to the equation with given boundary conditions. We will later use this result to prove an energy expansion of first and second order for minimizers. In particular this shows that it is the unique minimizer with respect to its boundary conditions (this is one of the main results in [12] which is used in [25] to compute a second order lower bound for the energy)

Let $u = u(x, y)$ be a function defined on $Q := \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ be defined as

$$u(x, y) := \arctan\left(\frac{y+1}{x+1}\right). \tag{4.2}$$

We start by introducing some notation: first we define B_R^+ to be $B_R \cap Q$. Set $\Gamma_x := \{(x, 0) : x > 0\}$ and $\Gamma_y =: \{(0, y) : y > 0\}$. Similarly for $R > 0$ we define $\Gamma_x^R := \Gamma_x \cap B_R$ and $\Gamma_y^R := \Gamma_y \cap B_R$. We finally set $\Gamma_0 := \Gamma_x \cup \Gamma_y$ and $\Gamma_R^+ := \{(x, y) \in \overline{Q} : |(x, y)| = R\}$.

Then we have the following result:

Theorem 4.1. *The function u defined by (4.2) satisfies the following equation:*

$$\begin{cases} \Delta u = 0 & \textit{in} \quad Q \\ \frac{\partial u}{\partial \nu} = \frac{1}{2} \sin 2u & \textit{on} \quad \Gamma_y \\ \frac{\partial u}{\partial \nu} = -\frac{1}{2} \sin 2u & \textit{on} \quad \Gamma_x, \end{cases} \tag{4.3}$$

where ν denotes the outward normal.

Proof. We have by a direct calculation:

$$\frac{\partial}{\partial y} \arctan\left(\frac{y+1}{x+1}\right) = \frac{x+1}{(x+1)^2 + (y+1)^2} \tag{4.4}$$

$$\frac{\partial}{\partial x} \arctan\left(\frac{y+1}{x+1}\right) = -\frac{y+1}{(x+1)^2 + (y+1)^2}. \tag{4.5}$$

Furthermore we have that

$$\frac{1}{2} \sin 2 \arctan\left(\frac{y+1}{x+1}\right) = \frac{(x+1)(y+1)}{(y+1)^2 + (x+1)^2}. \tag{4.6}$$

The conclusion follows easily from this.

□

Before we continue we recall the definition of a *local minimizer* from [12] (see Definition 1.1):

Definition 4.2 (Local minimizer)**.** *Let $R > 0$ and let $E : H^1\left(B_R^+\right) \to \mathbb{R}$ be an energy functional. A function $u \in H^1\left(B_R^+\right)$ satisfying $0 < u < \frac{\pi}{2}$ is called a local minimizer if*

$$E(u) \leq E(u + \varphi), \tag{4.7}$$

for all functions φ with compact support in $B_R^+ \cup \Gamma_R^0$ such that $0 < u + \varphi < \frac{\pi}{2}$.

Our next step is to show that the function u defined above is a local minimizer of the energy: we will follow a similar approach to Cabrè and Solà-Morales [12] (see Lemma 3.5 there for the analogous result), using a sliding method to prove uniqueness of a solution given certain boundary conditions. It must be noted that in their paper [12] Cabrè and Solà-Morales show the local minimality for any layer solution, i.e. any solution which is monotone in the direction parallel to the boundary. In our case it could be an interesting result to define a similar notion of layer solution and then show uniqueness of such layer solutions, which we will not do here. We will instead prove the uniqueness result for the function u alone. The limits for u in the two directions are 0 and $\frac{\pi}{2}$, so local minimality of u amounts will follow if we can show that, for all $R > 0$ u, is the unique solution w to the following equation:

$$\begin{cases} \Delta w = 0 & \text{in} \quad B_R^+ \\ 0 \leq w \leq \pi/2 & \text{in} \quad \overline{B_R^+} \\ \frac{\partial w}{\partial \nu} = \frac{1}{2} \sin 2w & \text{on} \quad \partial Q \cap \{x = 0\} \cap B_R \\ \frac{\partial w}{\partial \nu} = -\frac{1}{2} \sin 2w & \text{on} \quad \partial Q \cap \{y = 0\} \cap B_R \\ w = u & \text{on } \partial B_R \cap Q. \end{cases} \tag{4.8}$$

We will show this by a sliding argument as in [12, Lemma 3.1], adapted to our case, which will require some modifications due to the presence of the angle. As preliminary results to our main result we will study the regularity of problem (4.8). These will be needed in the proof, but are of independent interest. Our main result is

Theorem 4.3 (Uniqueness). *Let u be defined as in (4.2). Then, for every $R > 0$,* *u is the unique weak solution of the problem*

$$\begin{cases} \Delta w = 0 & \textit{in} \quad B_R^+ \\ 0 \leq w \leq \pi/2 & \textit{in} \quad \overline{B_R^+} \\ \frac{\partial u}{\partial \nu} = \frac{1}{2}\sin 2w & \textit{on} \quad \partial Q \cap \{x=0\} \cap B_R \\ \frac{\partial u}{\partial \nu} = -\frac{1}{2}\sin 2w & \textit{on} \quad \partial Q \cap \{y=0\} \cap B_R \\ w = u & \textit{on}\ \partial B_R \cap Q. \end{cases} \tag{4.9}$$

We need some regularity result for solutions of (4.3). More precisely we want to study, for all $R > 0$ the regularity of weak solutions $u \in H^1(Q \cap B_R(0))$ in the quadrant $Q := \{(x,y) : x, y > 0\}$, i.e solutions of

$$\int_Q \nabla u \cdot \nabla \varphi d\mathbf{x} - \frac{1}{2}\int_{\Gamma_y} (\sin 2u)\, \varphi d\mathcal{H}^1 + \frac{1}{2}\int_{\Gamma_x} (\sin 2u)\, \varphi d\mathcal{H}^1 = 0, \tag{4.10}$$

for all $\varphi \in C^\infty\left(\overline{Q}\right)$ which satisfy $\operatorname{supp}\varphi \subset \overline{B_R \cap Q}$, where $\mathbf{x} = (x,y)$, and Γ_y and Γ_x are the y and x half-axes respectively. These exist since they arise as blow-up in a corner of solutions of the Euler-Lagrange equation for a variational problem in a rectangle. By a result of Cabré and Solà-Morales [12] and by interior regularity for harmonic functions we can show that any weak solution of this problem belongs to $C^\infty\left(\overline{Q} \setminus \{0\}\right)$. If we consider a rectangle of the form $R_{a,b} := (0,a) \times (0,b)$ we can prove using a result by Jerison and Kenig [29] that a solution w is in $H^{3/2}(R_{a,b})$, which in particular implies that $w \in C^{0,1/2}\left(\overline{R_{a,b}}\right)$. Define a function v for $x, y \geq 0$ as:

$$v(x,y) := \int_0^x \int_0^y w(s,t)\, dt ds. \tag{4.11}$$

We clearly have that

$$v_x(x,y) = \int_0^y u(x,t)\, dt \quad \text{and} \quad v_y(x,y) = \int_0^x u(s,y)\, ds, \tag{4.12}$$

and consequently, using integration by parts, we have that

$$v_{xx}\left(x,y\right)=\int_0^y u_x\left(x,t\right)dt=\int_0^y\int_0^x u_{xx}\left(s,t\right)dsdt+\int_0^y u_x\left(0,t\right)dt \tag{4.13}$$

and

$$v_{yy}\left(x,y\right)=\int_0^x u_y\left(x,t\right)dt=\int_0^x\int_0^y u_{yy}\left(s,t\right)dtds+\int_0^x u_y\left(s,0\right)ds. \tag{4.14}$$

Adding the last two equations and using the fact that u is harmonic, we obtain

$$\begin{aligned}\Delta v\left(x,y\right)&=\int_0^y u_x\left(0,t\right)dt+\int_0^x u_y\left(s,0\right)ds\\&=-\int_0^y\frac{1}{2}\sin 2u\left(0,t\right)dt+\int_0^x\frac{1}{2}\sin 2u\left(s,0\right)ds.\end{aligned} \tag{4.15}$$

We can now easily compute that $\left(\Delta v\right)_{xy}=0$ and $\left(\Delta v\right)_{xx}=\cos 2u\left(x,0\right)u_x\left(x,0\right)$. It can be easily seen that u is smooth away from the corners, and hence ∇u is well defined at any point of the boundary which is not a corner, for any rectangle $\mathcal{R}$ with a vertex in the origin and sides parallel to the coordinate axes, and is point-wise bounded through its non-tangential maximal function. Then we can apply Theorem 2 in [29] to conclude that the gradient is in $L^2\left(\partial\mathcal{R}\right)$, hence in particular we conclude that $\left(\Delta v\right)_{xx}\in L^2\left(\mathcal{R}\right)$. In the same way we conclude that $\left(\Delta v\right)_{yy}\in L^2\left(\mathcal{R}\right)$. This means that $\Delta v\in H^2\left(\mathcal{R}\right)$.

Now consider a smooth cut-off function ψ with support in $R_{a,b}$ that is equal to 1 on a ball of radius (e.g.) $\frac{1}{4}\min\{a,b\}$ centred at 0: we can conclude from Hell and Ostermann [23, Proposition 1] (which we report below for the ease of the reader as Theorem 4.4 below) that $v\in H^4\left(\overline{R_{a,b}}\right)$. Hence we conclude that $w\in H^2\left(B_{1/4\min\{a,b\}}\left(0\right)\right)$. Since we can do this for all a,b we conclude that $w\in H^2\left(B_R\left(0\right)\right)$ for all $R>0$.

Theorem 4.4. *[Hell and Ostermann [23, Proposition 1]] Let Ω be a rectangle, $k\geq 1$. For a function $f\in\overline{\mathcal{C}}\left(\Omega\right)$ and $j\in\{1,\dots,k\}$ we set*

$$C_j f = \sum_{i=1}^{j} (-1)^{i+1} \partial_x^{2j-2i} \partial_y^{2i-2} f.$$

We then define the compatibility conditions to be:

$$C_j f_{\big|V} = 0 \quad \text{for all } j = 1, \ldots, k, \tag{4.16}$$

where V denotes the set consisting of the four vertices. Then we have that for a given $f \in H^{2k}(\Omega)$ the solution to the problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.17}$$

is in $H^{2k+2}(\Omega)$ if and only if the compatibility conditions (4.16) hold.

Remark 1. Observe that for $k = 1$ the compatibility conditions amount to f being 0 in all corners.

We can then say that any weak solution is a strong solution to:

$$\begin{cases} \Delta u = 0 & \text{in} \quad Q \\ -\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \nu} = \frac{1}{2}\sin 2u & \text{on} \quad \Gamma_y \\ -\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \nu} = -\frac{1}{2}\sin 2u & \text{on} \quad \Gamma_x. \end{cases} \tag{4.18}$$

Before we proceed we show that indeed $0 < w < \frac{\pi}{2}$ in $\overline{Q \cap B_R}$. Assume that $w(x_0, y_0) \in \{0, \frac{\pi}{2}\}$ for some $(x_0, y_0) \in \overline{Q \cap B_R}$. By the maximum principle this cannot happen in the interior $Q \cap B_R$. Furthermore, since $w = u$ on Γ_R^+, and $0 < u < \frac{\pi}{2}$, we also have that $(x_0, y_0) \notin \Gamma_R^+$. If (x_0, y_0) is on the boundary but not in the origin, we conclude from the boundary condition that the normal derivative must vanish, but this is impossible for the Hopf boundary lemma. So the only possibility is that $(x_0, y_0) = (0, 0)$. At such a point we need a different strategy: define a function $\hat{w}$ in the upper-half space, via a conformal transformation, as:

$$\hat{w}\left(x^2-y^2,2xy\right):=w\left(x,y\right). \tag{4.19}$$

Then $\hat{w}$ satisfies the following PDE:

$$\begin{cases}\Delta\hat{w}=0 & \text{in } \mathbb{R}^2_+\\ \frac{\partial\hat{w}}{\partial\nu}=\frac{\operatorname{sgn}x}{\sqrt{|x|}}\sin 2\hat{w}\left(x,0\right) & \text{on } \partial\mathbb{R}^2_+.\end{cases} \tag{4.20}$$

Now, since $\hat{w}$ is locally in every Hölder space with exponent in $(0,1/2)$ we conclude that its normal derivative must be in every $L^p\left(\mathbb{R}\right)$, and we conclude with the help of the Hilbert transform[2] that $\hat{w}\in W^{1,p}_{loc}$, hence locally in C^α for all $\alpha\in(0,1)$. So we conclude that the normal derivative is in C^γ for all $\gamma\in(0,1/2)$: then (see for example [40, Theorem 2]) we have that $\hat{w}\in C^{1,\tilde{\gamma}}$ for some $\tilde{\gamma}\in(0,1/2)$. In the interior we have C^2 (indeed C^∞) regularity, so we can apply Hopf's lemma to conclude that $\frac{\partial\hat{w}(0,0)}{\partial\nu}<0$ if $u\left(0,0\right)=0$ (or $\frac{\partial\hat{w}}{\partial\nu}>0$ if $u\left(0,0\right)=\pi/2$). We will get a contradiction if we can show that this derivative is equal to 0. For brevity let $f\left(x\right)=\sin 2\hat{w}\left(x,0\right)$ and $g=\frac{\operatorname{sgn}x}{\sqrt{|x|}}f\left(x\right)$. The function g is Hölder continuous, so the limit $\lim_{x\to0}g\left(x\right)$ must exists on both sides and be equal, call it L. If $L>0$ we have a contradiction considering the limit from the left side, since there the function is always non-positive, because $f\geq0$. Analogously we conclude that L cannot be negative. Hence $L=0$, which implies $\frac{\partial\hat{w}(0,0)}{\partial\nu}=0$, which is a contradiction to the Hopf Lemma. Hence we conclude that $u\left(0,0\right)\notin\{0,\pi/2\}$.

As next step we prove that all inwardly-pointing derivatives exist at the origin and we give an explicit expression for them:

Theorem 4.5. *Let $s=(s_1,s_2)$ be an inwardly pointing direction, i.e. $s_1,s_2>0$, and let u be a solution of (4.18). Then the directional derivative $\frac{\partial u(0,0)}{\partial s}$ exists*

[2]This is done observing that the boundary value problem $\frac{\partial w}{\partial\nu}=f$ is equivalent to $H\left(w_\xi\right)=f$, where H is the Hilbert transform. The Hilbert transform is a bounded linear operator on L^p for $1<p<\infty$ and its inverse is $-H$. Then we see that $w_\xi=H\left(f\right)\in L^p\left(\mathbb{R}\right)$ if $f\in L^p\left(\mathbb{R}\right)$

and satisfies:

$$\frac{\partial u\left(0,0\right)}{\partial s}=\left(-\frac{1}{2}\sin 2u\left(0,0\right),\frac{1}{2}\sin 2u\left(0,0\right)\right)\cdot\left(s_1,s_2\right). \tag{4.21}$$

Proof. We will show this by expressing a solution of (4.18) by means of Green's functions. Consider a positive radius $R<1/2$ (this only for convenience later, any $R>0$ would work as fine) and let φ be a smooth cutoff function which is equal to 1 in the ball $B_R\left(0\right)$, and equal to 0 outside the ball $B_{2R}\left(0\right)$. Then the function $v=u\varphi$ has compact support, is smooth in Q and its Laplacian satisfies:

$$\begin{cases}\Delta v=0 & \text{in } B_R^+\\ \Delta v=2\nabla u\cdot\nabla\varphi+u\Delta\varphi & \text{in } B_{2R}^+\setminus B_R^+\\ \Delta v=0 & \text{in } Q\setminus B_{2R}^+.\end{cases} \tag{4.22}$$

By the product rule we get $\frac{\partial v}{\partial\nu}=\frac{\partial u}{\partial\nu}\varphi+u\frac{\partial\varphi}{\partial\nu}$ and so

$$\begin{cases}\frac{\partial v}{\partial\nu}=\frac{\partial u}{\partial\nu} & \text{in } B_R^+\\ \frac{\partial v}{\partial\nu}=\frac{\partial u}{\partial\nu}\varphi+u\frac{\partial\varphi}{\partial\nu} & \text{in } B_{2R}^+\setminus B_R^+\\ \frac{\partial v}{\partial\nu}=0 & \text{in } Q\setminus B_{2R}^+.\end{cases} \tag{4.23}$$

From this, and using the Neumann boundary conditions for u we conclude that v satisfies a boundary value problem with Neumann boundary conditions, such that the Neumann boundary condition on each half-axis is a C^α function (for all $\alpha\in(0,1)$) with compact support. We will prove the result for v, which obviously will imply it for u, since they coincide in a neighbourhood of the origin. Consider the function $\tilde{v}=v+\frac{1}{2}\sin 2u\left(0,0\right)\left(x-y\right)\varphi$. Then the conclusion will be proved if we can show that $\nabla\tilde{v}\left(x,y\right)\to 0$ for $\left(x,y\right)\to\left(0,0\right)$ in Q. Let $G\left(x,y,\xi,\eta\right)$ be the Green function for the Neumann problem in the quarter-space Q: from [46] (see 1.1 on page 2) we have that such Green function has the form (modulo a multiplicative constant which doesn't play any role for us):

$$
\begin{aligned}
G(x,y,\xi,\eta) =& \frac{1}{2}\log\left((\xi-x)^2+(\eta-y)^2\right)+\frac{1}{2}\log\left((\xi+x)^2+(\eta-y)^2\right)\\
&+\frac{1}{2}\log\left((\xi-x)^2+(\eta+y)^2\right)+\frac{1}{2}\log\left((\xi+x)^2+(\eta+y)^2\right).
\end{aligned}
\tag{4.24}
$$

This function satisfies (see [46, Theorem 1]) the following conditions:

$$
\frac{\partial G(x,y,0,\eta)}{\partial \xi}=\frac{\partial G(x,y,\xi,0)}{\partial \eta}=0. \tag{4.25}
$$

By Green's theorem we get then that $\tilde{v}$ can be expressed in term of the Green's function as follows (where $\xi,\eta>0$):

$$
\begin{aligned}
\tilde{v}(\xi,\eta) =& \int_0^\infty \frac{\partial \tilde{v}}{\partial \nu}(0,y)\,G(0,y,\xi,\eta)\,dy+\int_0^\infty \frac{\partial \tilde{v}}{\partial \nu}(x,0)\,G(x,0,\xi,\eta)\,dx\\
&+\int_{B_{2R}^+\setminus B_R^+} G(x,y,\xi,\eta)\,\Delta\tilde{v}(x,y)\,dxdy.
\end{aligned}
\tag{4.26}
$$

By the properties of u and the definition of $\tilde{v}$ we have that for all $\alpha\in(0,1)$:

$$
\left|\frac{\partial \tilde{v}}{\partial \nu}(x,0)\right|\le C|x|^\alpha,\quad \left|\frac{\partial \tilde{v}}{\partial \nu}(0,y)\right|\le C|y|^\alpha. \tag{4.27}
$$

We now will show that $\frac{\partial \tilde{v}}{\partial \xi}(\xi,\eta)\to 0$ for $(\xi,\eta)\to(0,0)$. The proof for $\frac{\partial \tilde{v}}{\partial \eta}$ is analogous and is left to the reader. Without loss of generality we can assume that $\xi^2+\eta^2<1$. We have:

$$
\begin{aligned}
\frac{\partial \tilde{v}}{\partial \xi}(\xi,\eta) =& \int_0^\infty \frac{\partial \tilde{v}}{\partial \nu}(0,y)\,\frac{\partial G(0,y,\xi,\eta)}{\partial \xi}dy+\int_0^\infty \frac{\partial \tilde{v}}{\partial \nu}(x,0)\,\frac{\partial G(x,0,\xi,\eta)}{\partial \xi}dx\\
&+\int_{B_{2R}^+\setminus B_R^+} \frac{\partial G(x,y,\xi,\eta)}{\partial \xi}\Delta\tilde{v}(x,y)\,dxdy.
\end{aligned}
\tag{4.28}
$$

The last term is easily seen to converge to 0 as $(\xi,\eta)\to(0,0)$, by dominated convergence and the properties of G. So we need to show that the other two terms converge to 0. We have that

$$\frac{\partial G(x,0,\xi,\eta)}{\partial \xi} = \frac{2(\xi - x)}{\eta^2 + (\xi - x)^2} + \frac{2(\xi + x)}{\eta^2 + (\xi + x)^2} \tag{4.29}$$

and

$$\frac{\partial G(0,y,\xi,\eta)}{\partial \xi} = \frac{2\xi}{\xi^2 + (\eta - y)^2} + \frac{2\xi}{\xi^2 + (\eta + y)^2}. \tag{4.30}$$

We have, by (4.23):

$$\int_0^\infty \frac{\partial \tilde{v}(0,y)}{\partial \nu} \frac{\partial G(0,y,\xi,\eta)}{\partial \xi} dy = \int_0^1 2\xi \frac{\partial \tilde{v}(0,y)}{\partial \nu} \left(\frac{1}{\xi^2 + (\eta - y)^2} + \frac{1}{\xi^2 + (\eta + y)^2} \right) dy.$$

We now have

$$\begin{aligned} &\left| \int_0^1 2\xi \frac{\partial \tilde{v}(0,y)}{\partial \nu} \left(\frac{1}{\xi^2 + (\eta - y)^2} + \frac{1}{\xi^2 + (\eta + y)^2} \right) dy \right| \\ &\leq \int_0^1 2\xi y^\alpha \left(\frac{1}{\xi^2 + (\eta - y)^2} + \frac{1}{\xi^2 + (\eta + y)^2} \right) dy. \end{aligned} \tag{4.31}$$

Now choose $\alpha = 1/2$, so the right-hand side becomes

$$\begin{aligned} 2\xi \int_0^1 \sqrt{y} \frac{\xi^2 + \eta^2 + y^2}{\left(\xi^2 + (\eta - y)^2\right)\left(\xi^2 + (\eta + y)^2\right)} dy = &\frac{1}{2} \frac{1+i}{\sqrt{2}} \sqrt{\xi - i\eta} \arctan\left(\frac{\frac{1+i}{\sqrt{2}}}{\sqrt{\xi - i\eta}} \right) \\ &- \frac{1}{2} \frac{1-i}{\sqrt{2}} \sqrt{\xi + i\eta} \arctan\left(\frac{-\frac{1-i}{\sqrt{2}}}{\sqrt{\xi + i\eta}} \right) \\ &+ \frac{1}{2} \frac{\xi + i\eta}{\eta - i\xi} \sqrt{\eta - i\xi} \arctan\left(\frac{1}{\sqrt{\eta - i\xi}} \right) \\ &+ \frac{1}{2} \frac{\xi - i\eta}{\eta + i\xi} \sqrt{\eta + i\xi} \arctan\left(\frac{1}{\sqrt{\eta + i\xi}} \right). \end{aligned} \tag{4.32}$$

Here we observe that the argument of the square root always belongs to the set $\mathbb{C} \setminus \mathbb{R} \setminus i\mathbb{R}$, which means that we can choose a branch cut for the square root along the negative real axis. We then have that the value of the square root can never be on the imaginary axis (and neither can therefore its inverse), so we can choose a branch cut for the arctan along such axis, e.g $\{it : |t| \geq 1\}$. Then all the functions are well-defined and single-valued for $\xi, \eta > 0$.

We now can conclude that this tends to 0 for $(\xi, \eta) \to 0$ observing that for complex w we have $\lim_{w\to 0} w \arctan(1/w) = 0$ and that $\left|\frac{\xi+i\eta}{\eta-i\xi}\right| = \left|\frac{\xi-i\eta}{\eta+i\xi}\right| = 1$. The first identity follows from the identity $\arctan(1/w) = \operatorname{arccot}(w)$ and the fact that $\lim_{w\to 0} \operatorname{arccot}(w) = \pi/2$.

For the other term we obtain again from (4.23) that

$$\int_0^\infty \frac{\partial \tilde{v}(x,0)}{\partial \nu} \frac{\partial G(x,0,\xi,\eta)}{\partial \xi} dx = \int_0^1 \frac{\partial \tilde{v}(x,0)}{\partial \nu} \left(\frac{\xi - x}{\eta^2 + (\xi - x)^2} + \frac{\xi + x}{\eta^2 + (\xi + x)^2} \right) dx.$$

We now have

$$\begin{aligned} &\left| \int_0^1 \frac{\partial \tilde{v}(x,0)}{\partial \nu} \left(\frac{\xi - x}{\eta^2 + (\xi - x)^2} + \frac{\xi + x}{\eta^2 + (\xi + x)^2} \right) dx \right| \\ &\leq \int_0^1 x^\alpha \left| \left(\frac{\xi - x}{\eta^2 + (\xi - x)^2} + \frac{\xi + x}{\eta^2 + (\xi + x)^2} \right) \right| dx. \end{aligned} \tag{4.33}$$

Again choose $\alpha = 1/2$, so the right-hand side becomes (we computed the integrals with the help of Mathematica):

$$2\xi\int_0^1 \sqrt{x}\frac{|\xi^2+\eta^2-x^2|}{\left(\eta^2+(\xi-x)^2\right)\left(\eta^2+(\xi+x)^2\right)}dx = -\frac{1}{2}\sqrt{-\xi-i\eta}\arctan\left(\frac{1}{\sqrt{-\xi-i\eta}}\right)$$
$$+\frac{1}{2}\sqrt{\xi-i\eta}\arctan\left(\frac{1}{\sqrt{\xi-i\eta}}\right)$$
$$-\frac{1}{2}\sqrt{-\xi+i\eta}\arctan\left(\frac{1}{\sqrt{-\xi+i\eta}}\right)$$
$$+\frac{1}{2}\sqrt{\xi+i\eta}\arctan\left(\frac{1}{\sqrt{\xi+i\eta}}\right). \tag{4.34}$$

We then conclude since $\lim_{w\to 0} w\arctan(1/w) = 0$ for $w \in \mathbb{C}$. We can now show the conclusion easily: take an inwardly pointing direction s: we want to show that the derivative $\frac{\partial\tilde{v}}{\partial s}$ (and consequently the same derivative for u) exists and is 0, so that for u we will have that

$$\frac{\partial u(0,0)}{\partial u} = \left(-\frac{1}{2}\sin 2u(0,0), \frac{1}{2}\sin 2u(0,0)\right)\cdot(s_1, s_2). \tag{4.35}$$

Let $I = [0,1]$ and define a function f on I as

$$f(t) = \begin{cases} 0 & \text{if } t = 0 \\ \nabla\tilde{v}(ts)\cdot s & \text{otherwise.} \end{cases} \tag{4.36}$$

This is seen to be continuous on I, since $\nabla\tilde{v}(ts) \to 0$ for $t \to 0$. Now by this and the mean value theorem we have that for a $\xi_t \in (0,t)$:

$$\frac{\tilde{v}(ts)-\tilde{v}(0)}{t} = f(\xi_t). \tag{4.37}$$

Now the right hand side tends to 0 when t (and hence ξ_t) tends to 0, since f is continuous at 0. This shows that the limit of difference quotients at 0 exists and is 0, which yields the conclusion.

□

We can now prove Theorem 4.3: we will do this by sliding the explicit solution u along the two coordinate axes and then using arguments very similar to those in [12, Lemma 3.1]. For this we need to study what happens to a solution when we slide. We will show that according to the direction in which we slide, we will obtain sub-/supersolutions. We have the following result:

Proposition 4.6. *Let u be defined as in (4.2). For $t > 0$, define u_t to be the sliding of u in the x direction, i.e. $u_t(x, y) := u(x + t, y)$. Analogously we define v_t as the sliding in the y direction. Then u_t and v_t satisfy the following equations:*

$$\begin{cases} \Delta u_t = 0 \quad \text{for } x, y > 0 \\ \frac{\partial u_t}{\partial \nu} = -\frac{1}{2}\sin 2u_t \quad \text{on} \quad \{y = 0\} \\ \frac{\partial u_t}{\partial \nu} = \frac{1}{1+t}\left(\frac{1}{2}\sin 2u_t\right) \quad \text{on} \quad \{x = 0\}. \end{cases} \tag{4.38}$$

$$\begin{cases} \Delta v_t = 0 \quad \text{for } x, y > 0 \\ \frac{\partial v_t}{\partial \nu} = -\frac{1}{1+t}\left(\frac{1}{2}\sin 2v_t\right) \quad \text{on} \quad \{y = 0\} \\ \frac{\partial v_t}{\partial \nu} = \frac{1}{2}\sin 2v_t \quad \text{on} \quad \{x = 0\}. \end{cases} \tag{4.39}$$

Proof. We present only the proof for u_t, since the proof for v_t is similar. We have by a direct calculation

$$\frac{\partial}{\partial y}\arctan\left(\frac{y+1}{x+1+t}\right) = \frac{x+1+t}{(x+1+t)^2 + (y+1)^2}. \tag{4.40}$$

$$\frac{\partial}{\partial x}\arctan\left(\frac{y+1}{x+1+t}\right) = -\frac{y+1}{(x+1+t)^2 + (y+1)^2}. \tag{4.41}$$

Furthermore we have that

$$\frac{1}{2}\sin 2\arctan\left(\frac{y+1}{x+t+1}\right) = \frac{(y+1)(x+1+t)}{((y+1))^2 + (x+1+t)^2}. \tag{4.42}$$

The conclusion follows easily from this. □

We can express the conclusion of Proposition 4.6 in terms of the original equation (4.3), showing that we obtain super-/subsolutions:

Corollary 4.7. *Let u_t and v_t be defined as in Proposition 4.6 above. Then they satisfy the following differential inequalities:*

$$\begin{cases} \Delta u_t = 0 \quad \textit{for } x, y > 0 \\ \frac{\partial u_t}{\partial \nu} + \frac{1}{2}\sin 2u_t = 0 \quad \textit{on} \quad \{y = 0\} \\ \frac{\partial u_t}{\partial \nu} - \frac{1}{2}\sin 2u_t > 0 \quad \textit{on} \quad \{x = 0\}. \end{cases} \tag{4.43}$$

$$\begin{cases} \Delta v_t = 0 \quad \textit{for } x, y > 0 \\ \frac{\partial v_t}{\partial \nu} + \frac{1}{2}\sin 2v_t < 0 \quad \textit{on} \quad \{y = 0\} \\ \frac{\partial v_t}{\partial \nu} - \frac{1}{2}\sin 2v_t = 0 \quad \textit{on} \quad \{x = 0\}. \end{cases} \tag{4.44}$$

Proof. Again, we only present the proof for u_t, since the proof for v_t follows in a completely analogous fashion. We have on $\{x = 0\}$, adding $\frac{1}{2}\sin 2u_t$ to both sides of the equation

$$\begin{aligned} \frac{\partial u_t}{\partial \nu} + \frac{1}{2}\sin 2u_t &= -\frac{1}{2}\sin 2u_t \cdot \frac{1}{1+t} + \frac{1}{2}\sin 2u_t \\ &= \frac{1}{2}\sin 2u_t \left(1 - \frac{1}{1+t}\right) \\ &= \frac{t}{t+1} \cdot \frac{1}{2}\sin 2u_t = \frac{t}{t+1} \cdot \frac{(t+1)(y+1)}{(t+1)^2 + (y+1)^2} > 0, \end{aligned} \tag{4.45}$$

which concludes the proof.

□

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. Let w be a solution of

$$
\begin{cases}
\Delta w = 0 & \text{in} \quad Q \cap B^R \\
0 < w < \pi/2 & \text{in} \quad \overline{Q \cap B_R} \\
\frac{\partial u}{\partial \nu} = \frac{1}{2}\sin 2w & \text{on} \quad \partial Q \cap \{x = 0\} \cap B_R := \Gamma_y \\
\frac{\partial u}{\partial \nu} = -\frac{1}{2}\sin 2w & \text{on} \quad \partial Q \cap \{y = 0\} \cap B_R := \Gamma_x \\
w = u & \quad \text{on } \partial B_R \cap Q := \Gamma_R^+.
\end{cases}
\tag{4.46}
$$

Let us consider now u_t: from its definition it is clear that $\|u\|_{L^\infty(Q\cap B_R(x,0))} \to 0$ as x tends to ∞, which means that u_t tends to 0 uniformly on $Q \cap B_R$ as $t \to \infty$. Since $0 < w < \frac{\pi}{2}$ and w is a continuous function on the compact set $\overline{Q \cap B_R}$, its minimum there will be strictly positive. Hence, for t large enough we have that $w > u_t$. We now want to prove that this inequality holds for all $t > 0$. We observe that if the inequality holds for some t_0, then it also trivially holds for all $t > t_0$, since $t \mapsto u_t\,(x, y)$ is decreasing in t.

Suppose, by contradiction, that

$$
s := \inf\{t > 0 : w > u_t \quad \text{in} \quad \overline{Q \cap B_R}\} > 0.
$$

We clearly have that $w \geq u_s$ in $\overline{Q \cap B_R}$. On Γ_R^+ we have that $w = u > u_s$, since $s > 0$, so in particular we see that $w \neq u_s$. Now, by the definition of s as infimum there exists a point $(x_0, y_0) \in \overline{Q \cap B_R} \backslash \Gamma_R^+$ such that $w\,(x_0, y_0) = u^s\,(x_0, y_0)$. That is we have that $w - u_s \geq 0$ and that $(w - u_s)\,(x_0, y_0) = 0$.

The function $w - u_s$ is harmonic, so by maximum principle we can exclude that $(x_0, y_0) \in Q \cap B_R$. If $x_0 > 0, y_0 = 0$ we have that $\frac{\partial(w-u_s)}{\partial \nu}\,(x_0, y_0) = 0$, since they both satisfy the same boundary condition, and they are equal at (x_0, y_0). But by Hopf's boundary lemma the normal derivative at such point must be strictly negative, which leads to a contradiction. If $x_0 = 0$ and $y_0 > 0$ we have that with $f\,(t) = \frac{1}{2}\sin 2t$ that

$$\frac{\partial\left(w-u_s\right)}{\partial\nu}=f\left(w\right)-f\left(u_s\right)\frac{1}{s+1}=\frac{s}{s+1}f\left(u_s\right)=s\frac{\partial u_s}{\partial\nu}=-s\frac{\partial u_s}{\partial x}>0.$$

But again by the Hopf boundary lemma we have that $\frac{\partial(w-u_s)}{\partial\nu}<0$, which is a contradiction. Hence the only possibility left is that $(x_0,y_0)=(0,0)$. We now want to show that also this cannot be the case. This will give us the desired contradiction, which proves that $s=0$. Hence $w\geq u$. Sliding in the y direction shows analogously that $w\leq u$, thus $u=w$.

So the only thing which is left to prove is that (x_0,y_0) cannot be the origin. Assume it is, then from the conditions $(w-u_s)(0,0)=0$ and $w-u_s\geq 0$ we conclude that $\frac{\partial(w-u_s)}{\partial v}\geq 0$ for any inwardly pointing direction v. Our aim is to show that in fact there exists one of such directions for which such derivative is strictly negative, thus giving a contradiction. By Theorem 4.5 we have that for any direction $v=(v_1,v_2)\,,v_1,v_2>0$:

$$\frac{\partial w\left(0,0\right)}{\partial v}=\left\langle\frac{1}{2}\left(-\sin 2w\left(0,0\right),\sin 2w\left(0,0\right)\right),v\right\rangle=\lambda\langle(-1,1)\rangle,$$

with $\lambda:=\frac{1}{2}\sin 2w\left(0,0\right)>0$, since $0<w\left(0,0\right)<\pi/2$. Even though the gradient at 0 might not exist, with a slight abuse of notation we denote by $\nabla w\left(0,0\right)$ the vector $\frac{1}{2}\left(-\sin 2w\left(0,0\right),\sin 2w\left(0,0\right)\right)$.

Let now $v:=(v_1,v_2)$, $v_i>0$ be an inwardly pointing direction. We want to choose v in such a way as to have $\frac{\partial(w-u_s)}{\partial v}\left(0,0\right)<0$. We have

$$\nabla w\left(0,0\right)=\lambda\left(-1,1\right)$$

and

$$\nabla u_t\left(0,0\right)=\frac{1}{1+\left(1+s\right)^2}\left(-1,1+s\right). \tag{4.47}$$

From this we can see that choosing $v = (1, 1)$ we obtain

$$\langle \nabla w (0,0) - \nabla u_s (0,0), v \rangle = -\frac{s}{1 + (1+s)^2} < 0, \tag{4.48}$$

which gives a contradiction. Hence $s = 0$. This shows that $w \geq u$. The opposite inequality is proved by sliding in the y direction.

$\square$

4.2 The energy of the explicit solution

In this section, as a preparation for the energy expansion results of Chapter 7, we compute the energy of the known explicit solution u in a quarter-ball B_R^+ of radius R, where u is defined as above as:

$$u(x,y) = \arctan\left(\frac{y+1}{x+1}\right) \tag{4.49}$$

and $B_R^+ := B_R \cap Q$. We have the following result:

Theorem 4.8. *Let u be defined as in (4.49). Then u has the following energy expansion in B_R^+:*

$$\int_{B_R^+} |\nabla u|^2 dx = \frac{\pi}{2} \log R + \frac{\pi}{2} \log 2 + G + O\left(\frac{\log R}{R}\right), \tag{4.50}$$

where G is Catalan's constant, for the Dirichlet part and

$$\frac{1}{2} \int_0^R \sin^2 u(x,0)\, dx + \frac{1}{2} \int_0^R \cos^2 u(0,y)\, dy = \frac{\pi}{2} + O\left(\frac{1}{R}\right) \tag{4.51}$$

for the penalty term. The constant G is defined as $G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$, and its numerical value approximately equals 0.916.

Proof. For the Dirichlet part we have

$$\int_{B_R^+} |\nabla u|^2 dx = \int_{B_R^+} \frac{1}{(x+1)^2 + (y+1)^2} dx$$
$$= \int_{B_R^+ + (1,1)} \frac{1}{x^2+y^2}. \tag{4.52}$$

This integrand is invariant with respect to the symmetry $(x, y) \mapsto (y, x)$, so we can rewrite it as

$$\int_{B_R^+ + (1,1)} \frac{1}{x^2+y^2} = 2 \int_{B_R^+ + (1,1) \cap \{x<y\}} \frac{1}{x^2+y^2}. \tag{4.53}$$

We want to use polar coordinates to find the expansion of this integral as $R \to +\infty$. For a fixed $R > 0$ we can describe the domain of integration in polar coordinates as (see Figure 4.2):

$$D_R := \left\{ (r, \theta) : \theta \in \left(\arctan\left(\frac{1}{1+R}\right), \frac{\pi}{4}\right), r \in (r_1(\theta), r_2(\theta)) \right\}. \tag{4.54}$$

The angle θ can vary between $\arctan\left(\frac{1}{1+R}\right)$ and $\pi/4$, while for every value of θ the radial variable assumes values in an interval $(r_1(\theta), r_2(\theta))$, where r_1 is the distance between the origin and the point where the line $y = \tan\theta x$ intersects $y = 1$ and r_2 the distance between the origin and the point where the same line intersects the set $\{x, y \geq 1, (x-1)^2 + (y-1)^2 = R^2\}$. To find $r_1(\theta)$ we notice that r_1 is the length of the hypotenuse of a right triangle whose other sides are 1 and $r_1 \cos\theta$. This gives then the following equation for $r_1 := r_1(\theta)$

$$r_1^2 = 1 + r_1^2 \cos^2\theta, \tag{4.55}$$

from which we deduce easily (by using $r_1 > 0$ by definition and $\sin\theta > 0$ since $\theta \in \left(0, \frac{\pi}{4}\right)$) that

$$r_1(\theta) = \frac{1}{\sin\theta}. \tag{4.56}$$

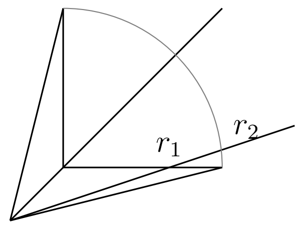


Figure 4.2: The construction of r_1 and r_2

To find r_2 we write a point on $y = \tan\theta x$ as $r_2(\theta)(\cos\theta, \sin\theta)$ and we impose the condition that it lies on the prescribed arc of circumference. Thus we get the following equation for $r_2 := r_2(\theta)$:

$$(r_2\cos\theta - 1)^2 + (r_2\sin\theta - 1)^2 = R^2. \tag{4.57}$$

Expanding and grouping terms with the same degree we can rewrite this as

$$r_2^2 - 2(\sin\theta + \cos\theta)\, r_2 + 2 - R^2 = 0. \tag{4.58}$$

This is a quadratic equation with coefficients $a = 1, b = -2(\sin\theta + \cos\theta)$, $c = 2 - R^2$. Since we are interested in the expansion for $R \to \infty$, we can assume without restrictions that $c < 0$. Since $\theta \in (0, \pi/4)$ we have furthermore that $b < 0$. We have that the discriminant Δ satisfies:

$$\frac{\Delta}{4} = R^2 - 1 + \sin 2\theta > 0, \tag{4.59}$$

hence the equation has two distinct real solutions. Since we have $a > 0, b < 0, c < 0$ by Descartes' rule we deduce that one solution is positive and the other is negative. We discard the negative solution, since $r_2 > 0$ by definition, so in the end we conclude that

$$r_2(\theta) = \cos\theta + \sin\theta + \sqrt{R^2 - 1 + \sin 2\theta}. \tag{4.60}$$

So we have that the integral that we need to compute is

$$\int_{B_R^+} |\nabla u|^2 = 2 \int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \int_{\frac{1}{\sin\theta}}^{\cos\theta+\sin\theta+\sqrt{R^2-1+\sin 2\theta}} \frac{1}{r} dr d\theta. \tag{4.61}$$

We have

$$\begin{aligned}
&\int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \int_{\frac{1}{\sin\theta}}^{\cos\theta+\sin\theta+\sqrt{R^2-1+\sin 2\theta}} \frac{1}{r} dr d\theta \\
&= \int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \log\left[\left(\cos\theta + \sin\theta + \sqrt{R^2-1+\sin\theta}\right)\sin\theta\right] \\
&= \int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \log\left(\cos\theta + \sin\theta + \sqrt{R^2-1+\sin 2\theta}\right) - \int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \log\sin\theta d\theta.
\end{aligned}$$

Let

$$I(R) := \int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \log f_R(\theta)\, d\theta \tag{4.62}$$

where

$$f_R(\theta) = \cos\theta + \sin\theta + \sqrt{R^2-1+\sin 2\theta}. \tag{4.63}$$

Recall that $\sin\theta, \cos\theta > 0$ and $\sin 2\theta > 0$ (since $\theta \in (0, \pi/4)$), so

$$\sqrt{R^2-1} < f_R(\theta) < 2 + R, \tag{4.64}$$

and since the logarithm is an increasing function we get that

$$\int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \frac{1}{2}\log\left(R^2-1\right) d\theta \le I(R) \le \int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \log(2+R)\, d\theta, \tag{4.65}$$

which means

$$\begin{aligned}
\frac{1}{2}\left(\frac{\pi}{4} - \arctan\left(\frac{1}{1+R}\right)\right)\log\left(R^2-1\right) &\le I(R) \\
&\le \left(\frac{\pi}{4} - \arctan\left(\frac{1}{1+R}\right)\right)\log(R+2).
\end{aligned}$$

We now observe that the two functions on the side differ by $O\left(\frac{1}{R}\right)$ as $R \to \infty$, and that the same is true for their difference with $\left(\frac{\pi}{4} - \arctan\left(\frac{1}{1+R}\right)\right)\log(R)$, so we have $I(R) = \frac{\pi}{4}\log(R) + O\left(\frac{\log R}{R}\right)$. For the remaining term we have:

$$
\begin{aligned}
-\int_{\arctan\left(\frac{1}{1+R}\right)}^{\frac{\pi}{4}} \log\sin\theta d\theta &= -\int_0^{\frac{\pi}{4}} \log\sin\theta d\theta + O\left(\frac{\log R}{R}\right) \\
&= \frac{\pi}{4}\log 2 + \frac{1}{2}G + O\left(\frac{\log R}{R}\right),
\end{aligned}
\tag{4.66}
$$

where G is Catalan's constant and we used the expression for $\int_0^{\frac{\pi}{4}} \log\sin\theta d\theta$ in the book by Gradshteyn and Ryzhik [22, 4.224,(2)]. We also have estimated the integral $\int_0^{\arctan\left(\frac{1}{1+R}\right)} \log\sin\theta d\theta$ for R large enough as:

$$
\begin{aligned}
\left|\int_0^{\arctan\left(\frac{1}{1+R}\right)} \log\sin\theta d\theta\right| &\le \int_0^{\arctan\left(\frac{1}{1+R}\right)} |\log\sin\theta| d\theta \\
&= \int_0^{\arctan\left(\frac{1}{1+R}\right)} \log\frac{1}{\sin\theta} d\theta \le -\int_0^{\arctan\left(\frac{1}{1+R}\right)} \log\frac{\theta}{2} d\theta \\
&= \left[\theta\left(\log\left(\frac{\theta}{2}\right) - 1\right)\right]_0^{\arctan\left(\frac{1}{1+R}\right)} = O\left(\frac{\log R}{R}\right),
\end{aligned}
\tag{4.67}
$$

where we have used that for θ small enough (which is true for R large enough) we have $\log\frac{1}{\sin\theta} \le -\log\frac{\theta}{2}$. This completes the proof of the first part.

For the boundary term we have that

$$
\begin{aligned}
\frac{1}{2}\int_0^R \sin^2 u(x,0)\, dx &= \frac{1}{2}\int_0^R \sin^2\arctan\left(\frac{1}{x+1}\right) dx \\
&= \arctan(R+1) - \arctan(1) = \frac{\pi}{4} + O\left(\frac{1}{R}\right).
\end{aligned}
\tag{4.68}
$$

The same also holds for the integral on the y axis, from which we obtain the conclusion. $\square$

Chapter 5

First order lower bounds in rectangles

In this chapter we will prove some results for critical points in a rectangle; we will focus on minimizers and critical points which have the same energy as minimizers up to a constant. This is needed to justify rigorously some results on the energy of C and S states whose proof was sketched in the final chapter of [34] and which we will more closely examine in Chapter 6.

The energy functionals that we will study in this section assume the form

$$\int_{\Omega} |\nabla u|^2 dx + \frac{1}{2\pi\varepsilon} \sum_{k=1}^{4} \int_{L_k} \sin^2 (u - \alpha_k)\, d\mathcal{H}^1, \tag{5.1}$$

where as before $L_k, k = 1, \ldots, 4$ denote the sides of the rectangle, $\alpha_i \in \{m\frac{\pi}{2} : m \in \mathbb{Z}\}$ and where $|\alpha_k - \alpha_j| = \frac{\pi}{2}$ if $k - j \equiv 1 \mod 4$. If τ_k is the tangent vector to L_k then $e^{i\alpha_k} = \tau_k$. We recall a useful observation we made on page 62: given a function $u \in H^1(\Omega)$ which is bounded, we can obtain a new function $u^* \in H^1(\Omega)$ such that $0 \leq u^* \leq \pi$ in a way that doesn't increase the energy. In particular if we start with a minimizer we obtain another minimizer.

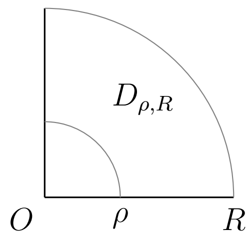


Figure 5.1: The quarter-annulus $D_{\rho,R}$.

5.1 First order lower bounds for the energy

In this section we will derive lower bounds for the energy of critical points of (5.1): we do this by adapting the method used by Struwe [51] and Kurzke [35]: the latter proved lower bounds for the energy on a domain with a smooth boundary (see also [34]). Our first result is the following lemma, which gives a lower bound for the energy of critical points on quarter-annuli:

Lemma 5.1. *Let $0 < \rho < R \leq R_0$ for some $R_0 > 0$ small enough. Let $d \in \mathbb{Z}$ and $0 < \delta \leq \frac{\pi}{4}$. Consider the portion of annulus $D_{\rho,R}$ around a corner (w.l.o.g. assumed to be 0) contained in Q with radii ρ and R, i.e. $D_{\rho,R} = \{(r,\theta) : r \in (\rho, R), \theta \in \left(0, \frac{\pi}{2}\right)\}$. Let $\theta_j = j\frac{\pi}{2}, d_j = jd$ for $j \in \{0,1\}$ and assume that $\left|u\left(re^{i\theta_j}\right) - \theta_j - d_j\pi\right| \leq \delta$ for all $r \in (\rho, R)$. Then there exists a constant C depending only on R_0 and d such that for every function u satisfying these hypotheses the energy is bounded below by*

$$E_\varepsilon\left(u; D_{\rho,R}\right) \geq \frac{2\left(\frac{\pi}{2} + d\pi\right)^2}{\pi} \log\frac{R}{\rho} - C\sqrt{\frac{\varepsilon}{\rho}}. \tag{5.2}$$

Proof. For ease of notation let $u_j(r) := u\left(re^{i\theta_j}\right)$. To estimate the Dirichlet part of the energy we use polar coordinates and disregard the radial derivative to get, by means of Hölder's inequality:

$$
\begin{aligned}
\int_{D_{\rho,R}} |\nabla u|^2 dx &\geq \int_\rho^R \frac{1}{r} \int_0^{\frac{\pi}{2}} \left|\frac{\partial u}{\partial \theta}\right|^2 d\theta dr \\
&\geq \int_\rho^R \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} \left|\frac{\partial u}{\partial \theta}\right| d\theta \right)^2 \frac{dr}{r} \geq \frac{2}{\pi} \int_\rho^R \frac{(u_1 - u_0)^2}{r} dr.
\end{aligned}
\tag{5.3}
$$

We can write $u_1 - u_0$ as $A - B$ where $A := \frac{\pi}{2} + d\pi$ and $B := u_0 - u_1 + \frac{\pi}{2} + d\pi$ and using the fact that $\sin^2(u_j - \theta_j - d_j\pi) \geq c\,(u_j - \theta_j - d_j\pi)^2$ for a constant depending on δ. Then we can write the following lower bound for the energy (where on the penalty term we use the inequality $a^2 + b^2 \geq \frac{1}{2}(a+b)^2$ for $a = u_0$ and $b = \frac{\pi}{2} + d\pi - u_1$) and where we denote $\gamma = \frac{\pi}{2}$

$$
\begin{aligned}
E_\varepsilon(u; D_{\rho,R}) &\geq \int_\rho^R \left(\frac{(A-B)^2}{\gamma r} + \frac{c}{2\varepsilon} B^2 \right) dr \geq \frac{1}{2} \int_\rho^R \frac{A^2}{r\gamma + \frac{2\varepsilon}{c}} dr \\
&= \int_\rho^R \frac{A^2}{\gamma r} dr + \int_\rho^R \left(\frac{A^2}{r\gamma + \frac{2\varepsilon}{c}} - \frac{A^2}{\gamma r} \right) dr \\
&= \frac{A^2}{\gamma} \log \frac{R}{\rho} - \int_\rho^R \frac{2\varepsilon A^2}{c\gamma^2 r^2 + 2\varepsilon\gamma r} dr,
\end{aligned}
\tag{5.4}
$$

where in the inequality in the first line we used the inequality $\alpha\,(A-B)^2 + \beta B^2 \geq \frac{A^2}{\frac{1}{\alpha} + \frac{1}{\beta}}$, with $\alpha = \frac{1}{\gamma r}$ and $\beta = \frac{c}{2\varepsilon}$. To obtain the conclusion we need to estimate the last term. We observe that we want a lower bound and that the integrand function is positive, hence we need to find an upper bound for it. We have:

$$
\begin{aligned}
\int_\rho^R \left(\frac{2A^2\varepsilon}{c\gamma^2 r^2 + 2\gamma\varepsilon r} \right) dr &\leq \int_\rho^R \left(\frac{2A^2\varepsilon}{c\gamma^2 r^2 + 2\gamma\varepsilon\rho} \right) dr \\
&\leq 2A^2\varepsilon \frac{\left| \arctan(KR) - \arctan(K\rho) \right|}{\sqrt{2c\gamma^3\varepsilon\rho}} \\
&\leq \frac{2\pi A^2 \varepsilon}{\sqrt{2c\gamma^3\varepsilon\rho}} \leq \frac{A^2}{\sqrt{2c\gamma^3}} \sqrt{\frac{\varepsilon}{\rho}} = C\sqrt{\frac{\varepsilon}{\rho}},
\end{aligned}
\tag{5.5}
$$

where $K = \sqrt{\frac{c}{2\gamma\varepsilon\rho}}$. This concludes the proof. $\square$

Remark 2. To make the proof easier to follow we have proved this Lemma for $\theta_j = j\frac{\pi}{2}, d_j = jd$ for $j \in \{0,1\}$ and assumed that $|u\left(re^{i\theta_j}\right) - \theta_j - d_j\pi| \leq \delta$ for all $r \in (\rho, R)$. It is then easy to see that we can analogously show the same lower bound if we assume $|u\left(re^{i\theta_j}\right) - \alpha_j - k_j\pi| \leq \delta$ for $\alpha_j \in \theta_j + \pi\mathbb{Z}$ and $k_j \in \mathbb{Z}$ and set d so that $\frac{\pi}{2} + d\pi = \alpha_1 + k_1\pi - \alpha_0 - k_0\pi$, since we can easily bring back this case to the one in the Lemma by subtracting an integer multiple of π (i.e. $\alpha_0 + k_0\pi$) from the function u, which does not change the energy.

5.1.1 Covering of the approximate vortex set

In the case of a smooth domain the *approximate vortex set* S_ε of a function u_ε is defined as

$$S_\varepsilon := \left\{x \in \partial\Omega : \sin^2\left(u_\varepsilon\left(x\right) - g\left(x\right)\right) \geq \frac{1}{4}\right\},$$

where g is a lift of the tangent vector field τ, i.e. a function $g : \partial\Omega \to \mathbb{R}$ such that (if we identify $\mathbb{R}^2$ and $\mathbb{C}$) we have $\tau = e^{ig}$. In a rectangle observe that g can be chosen constant equal to α_k on each side with a jump of $\pm\frac{\pi}{2}$ in each corner. In the case of a rectangle we define S_ε analogously but we make a slight modification: since we have some special points, namely the corners, where the function g has a jump, we include them by default in the set, and we say that a point which is not a corner is in the approximate vortex set precisely when it satisfies the condition in (5.6) So if $w_i, i = 1, \dots 4$ denote the corners of Ω we define the set S_ε for a rectangle Ω as

$$S_\varepsilon := \{w_i\}_{i=1}^4 \bigcup \left\{x \in \partial\Omega \setminus \{w_i\}_{i=1}^4 : \sin^2\left(u_\varepsilon\left(x\right) - g\left(x\right)\right) \geq \frac{1}{4}\right\}, \tag{5.6}$$

where g is constant and equal to α_k on each side L_k. By the definition of S_ε we have this obvious corollary:

Corollary 5.2. *If a point x is not in S_ε then we have that x lies on one side L_i (i.e. it is not a corner) and satisfies $\sin^2\left(u_\varepsilon\left(x\right) - \alpha_i\right) < \frac{1}{4}$. Therefore there exist constants $0 < \delta < \frac{\pi}{4}, c > 0$ such that for each $x \notin S_\varepsilon$ there exists a unique*

$k_\varepsilon^x \in \mathbb{Z}$ such that $|u_\varepsilon(x) - \alpha_i - \pi k_\varepsilon^x| \leq \delta$ and $\sin^2(u_\varepsilon - \alpha_i) \geq c|u_\varepsilon - \alpha_i - \pi k_\varepsilon^x|^2$. We also have that k_ε^x is constant on each connected component of $\partial\Omega \setminus S_\varepsilon$.

Proof. The fact that x lies on one side L_i and that $\sin^2(u_\varepsilon(x) - \alpha_i) < \frac{1}{4}$ follows from the definition of S_ε, since we included the corner points in the set S_ε. The second claim follows from this inequality and the properties of the function $\sin^2(\cdot - \alpha_i)$ near its zeroes. The uniqueness of k_ε^x is obvious. The continuity of u_ε (see Lemma 5.3 below) then implies that k_ε^x is constant on any connected component of $\partial\Omega \setminus S_\varepsilon$. $\square$

Lemma 5.3. *Let u be a critical point of E_ε. Then u is continuous on $\overline{\Omega}$. Therefore u is bounded on $\overline{\Omega}$. Furthermore we have that $u \in H^2(\Omega)$.*

Proof. Continuity away from the corners is easy to show by standard methods using the boundary condition (for example we can use [12, Lemma 2.3] for the locally rescaled equation – indeed this can be used to show C^2 smoothness up to the boundary away from corners); in the corner we can prove continuity as we did in the proof of Theorem 4.3, just before (4.11): the argument is the same, since in the corner the equation satisfied by a critical point is a rescaled version of the one there (in that context we also had upper and lower bounds, and a Dirichlet condition, but they are only used in the proof of uniqueness and play no role in the proof of continuity/regularity, which can therefore be carried out in the same way in this case too). Fix now $\rho > 0$ small enough; then outside of 4 balls of radius ρ centred in the corners we can use interior regularity and difference quotients near the boundary to prove H^2-regularity (in the same way as in [34, Proposition 2.5]). In each ball $B_\rho^+(z_k)$ around a corner z_k we can use the same arguments we employed just before Theorem 4.4 to conclude that $u \in H^2\left(B_\rho^+(z_j)\right)$. Putting everything together we get the conclusion. $\square$

We now show that the approximate vortex set S_ε can be covered by a finite number of ε balls, whose number is uniformly bounded in ε. We know this for a sufficiently smooth domain (say C^2, as shown in [34, 35]), and the proof is very similar, however we will have to slightly adapt that construction to account for the corners. We first observe that by doubling the radius if necessary we can cover the boundary with balls of only two kinds:

- Balls centred at a boundary point which is not a corner and whose intersection with the rectangle is a half-disk;
- Balls centred in a corner.

We need an equivalent of [35, Proposition 3.5] for corner points:

Proposition 5.4. *Let $\rho < \min\{a,b\}$, where a,b are the side-lengths, and let z_0 be a corner. Then for $\Gamma_\rho := \partial\Omega \cap B_\rho(z_0)$ and for any stationary point u of E_ε we have:*

$$\frac{1}{2\varepsilon}\int_{\Gamma_\rho} \sin^2(u-g)\, d\mathcal{H}^1 \leq A_{\varepsilon,u,z_0}(\rho), \tag{5.7}$$

where

$$A_{\varepsilon,u,z_0}(\rho) := \rho\int_{\Omega\cap\partial B_\rho(z_0)} |\nabla u|^2 + \frac{\rho}{\varepsilon}\int_{\partial\Omega\cap\partial B_\rho(z_0)} \sin^2(u-g)\, d\mathcal{H}^0. \tag{5.8}$$

Proof. Without loss of generality we can assume $z_0 = 0$. We follow the proof given by [35, Proposition 3.5]. We need to choose a smooth vector field Z which is tangential along the sides: we can simply take $Z = z$. We can follow the same calculations in that proof to get, using Pohozhaev identity (where we set $\omega_\rho := \Omega \cap B_\rho$ and denote by Γ_0^ρ and Γ_1^ρ the part of the two sides contained in B_ρ (so $\Gamma_\rho = \Gamma_0^\rho \cup \Gamma_1^\rho$) and $\beta_\rho := \partial B_\rho \cap \Omega$

$$\frac{1}{2}\int_{\partial\omega_\rho} (z\cdot\nu)|\nabla u|^2 = \int_{\partial\omega_\rho} \frac{\partial u}{\partial\nu} z\cdot\nabla u = \int_{\omega_\rho} \nabla u\cdot\nabla(z\cdot\nabla u). \tag{5.9}$$

We then get

$$\frac{1}{2}\int_{\partial\omega_\rho} (z\cdot\nu)|\nabla u|^2 = \frac{\rho}{2}\int_{\beta_\rho} |\nabla u|^2 d\mathcal{H}^1, \tag{5.10}$$

and

$$\int_{\partial\omega_\rho} \frac{\partial u}{\partial\nu} z\cdot\nabla u = \rho\int_{\beta_\rho}\left|\frac{\partial u}{\partial\nu}\right|^2 + \int_{\Gamma_0^\rho\cup\Gamma_1^\rho}\frac{\partial u}{\partial\nu} z\cdot\nabla u d\mathcal{H}^1. \tag{5.11}$$

Using that $z\cdot\nabla u = (z\cdot\tau)\frac{\partial u}{\partial\tau}$ and the fact that u, being a critical point, solves the Euler-Lagrange equation we get:

$$\int_{\Gamma_0^\rho\cup\Gamma_1^\rho}\frac{\partial u}{\partial\nu} z\cdot\nabla u d\mathcal{H}^1 = -\frac{1}{2\varepsilon}\int_{\gamma_0^\rho\cup\Gamma_1^\rho}\sin 2(u-g)\frac{\partial u}{\partial\tau}(z\cdot\tau)\, d\mathcal{H}^1 \tag{5.12}$$

We choose as tangent vector Z the vector z, so that $z\cdot\tau = |z|$, and we parametrize Γ_ρ with an arc-length parametrization φ such that $\varphi(0) = 0$. Then we have that $\frac{\partial}{\partial\tau}\varphi = \operatorname{sgn}(s)\frac{d}{ds}(s)$, so that the integral becomes (where $\tilde{u},\tilde{g}$ are defined accordingly):

$$\int_{-\rho}^{\rho}\sin 2(\tilde{u}(s)-\tilde{g}(s))\frac{d\tilde{u(s)}}{ds}\operatorname{sgn}(s)|s|ds. \tag{5.13}$$

Splitting the integral on the intervals $(-\rho,0)$ and $(0,\rho)$, and noticing that on both sides $d\tilde{g}/ds = 0$, we can rewrite this as (notice that $\operatorname{sgn}(s)|s| = s$):

$$-\frac{1}{2\varepsilon}\left(\int_{-\rho}^{0}\sin 2(\tilde{u}-\tilde{g})\frac{d(\tilde{u}-\tilde{g})}{ds}sds + \int_0^\rho \sin 2(\tilde{u}-\tilde{g})\frac{d(\tilde{u}-\tilde{g})}{ds}sds\right). \tag{5.14}$$

We observe that on both intervals the function φ is absolutely continuous and that $\sin 2(\tilde{u}-\tilde{g})\frac{d(\tilde{u-g})}{ds}\in L^1$, so we can integrate by parts and obtain that this is equal to:

$$\begin{aligned}&-\frac{1}{2\varepsilon}\left[s\sin^2(\tilde{u}-\tilde{g})(s)\right]_{-\rho}^{0} + \frac{1}{2\varepsilon}\int_{-\rho}^{0}\sin^2(\tilde{u}-\tilde{g})\,ds\\ &-\frac{1}{2\varepsilon}\left[s\sin^2(\tilde{u}-\tilde{g})(s)\right]_0^\rho + \frac{1}{2\varepsilon}\int_0^\rho \sin^2(\tilde{u}-\tilde{g})\,ds.\end{aligned} \tag{5.15}$$

We observe that the integral evaluation in the integration by parts at 0 is 0 independently of how we choose to define $\tilde{g}$ at such point, so we can finally conclude that this is equal to:

$$-\frac{\rho}{2\varepsilon}\int_{\partial\Omega\cap\partial B_\rho}\sin^2(u-g)\,d\mathcal{H}^0+\frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin^2(u-g)\,d\mathcal{H}^1 \tag{5.16}$$

Combining what we have obtained so far we get that:

$$\begin{aligned}\frac{1}{2\varepsilon}\int_{\Gamma_0^\rho\cup\Gamma_1^\rho}\sin^2(u-g)\,d\mathcal{H}^1 &= \frac{\rho}{2\varepsilon}\int_{\partial\Omega\cap\partial\beta_\rho}\sin^2(u-g)\,d\mathcal{H}^0+\frac{\rho}{2}\int_{\beta_\rho}|\nabla u|^2 d\mathcal{H}^1\\ &\quad-\rho\int_{\beta_\rho}\left|\frac{\partial u}{\partial\nu}\right|^2 d\mathcal{H}^1\\ &\leq A(\rho)\,.\end{aligned} \tag{5.17}$$

This concludes the proof. ☐

By arguing in the same way (indeed with an easier proof because now we do not have to worry about the jump of g in the corner) we can obtain the following

Proposition 5.5. *Let z_0 be a point on a side L_k and let $\rho>0$ be such that $\Gamma_\rho:=\partial\Omega\cap B_\rho(z_0)$ lies entirely on the side L_k. Then for for any stationary point u of E_ε we have:*

$$\frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin^2(u-g)\,d\mathcal{H}^1\leq A_{\varepsilon,u,z_0}(\rho)\,, \tag{5.18}$$

where

$$A_{\varepsilon,u,z_0}(\rho):=\rho\int_{\Omega\cap\partial B_\rho(z_0)}|\nabla u|^2+\frac{\rho}{\varepsilon}\int_{\partial\Omega\cap\partial B_\rho(z_0)}\sin^2(u-g)\,d\mathcal{H}^0. \tag{5.19}$$

We now need an equivalent of [35, Lemma 3.7], both on flat parts and in the corners: this is proved in the same way as it is done there. For the convenience of the reader we report the statement and the (easy) proof:

Lemma 5.6. *Let u_ε be a sequence of stationary points, satisfying the energy bound $E_\varepsilon(u_\varepsilon) \le M|\log\varepsilon|$. Then, for every $z_0 \in \partial\Omega$ which is either a corner or a point on a side at a distance of at least ε^{θ_2} from the vertices, the function A defined in (5.8) satisfies for every $0 < \theta_2 < \theta_1 < 1$*

$$\inf_{\varepsilon^{\theta_1} \le \rho \le \varepsilon^{\theta_2}} A(\rho) \le \frac{1}{|\log\varepsilon|} \frac{2}{\theta_1 - \theta_2} E_\varepsilon\left(u_\varepsilon; \Omega \cap B_{\varepsilon^{\theta_2}}(z_0)\right) \le \frac{2M}{\theta_1 - \theta_2}. \tag{5.20}$$

Proof. We have

$$\begin{aligned} M\log\frac{1}{\varepsilon} &\ge E_\varepsilon\left(u_\varepsilon; \Omega \cap B_{\varepsilon^{\theta_2}}(z_0)\right) \ge \frac{1}{2}\int_{\varepsilon^{\theta_1}}^{\varepsilon^{\theta_2}} \frac{A(\rho)}{\rho} \\ &\ge \frac{1}{2}\inf A \log\frac{1}{\varepsilon^{\theta_1-\theta_2}} = \frac{\theta_1-\theta_2}{2}\log\frac{1}{\varepsilon}. \end{aligned}$$

□

We now show a result similar to [35, Proposition 3.6] which will be needed in the proof of Proposition 5.8 below:

Proposition 5.7. *There is a constant $\gamma > 0$ depending on Ω such that for every $z_0 \in \partial\Omega, \varepsilon > 0, \rho < \varepsilon^{3/4}$ such that $\Gamma_\rho(z_0)$ lies entirely on one side L_k and every stationary points u of E_ε satisfying $A(\rho) < \gamma$, there holds:*

$$\sup_{\Gamma_{\rho/2}(z_0)} \sin^2(u - \alpha_k) < \frac{1}{4} \tag{5.21}$$

Then we also have that there exists a constant $C > 0$ depending only on Ω such that

$$\frac{1}{2\varepsilon}\int_{\Gamma_\rho(z_0)} \sin^2(u - \alpha_k)\, d\mathcal{H}^1 \le C \tag{5.22}$$

Proof. We follow the proof of [35, Proposition 3.6]. We notice that if γ as above exists, then (5.22) will follow by Proposition 5.5 and the bound $A(\rho) < \gamma$. Assume then by contradiction that no such γ exists such that (5.21) is satisfied: then for all values of $\gamma, \varepsilon_0 > 0$ there exists $z_0 \in \partial\Omega$, $\varepsilon > 0$ and $\rho < \varepsilon^{3/4}$ such that $\Gamma_\rho(z_0)$ lies entirely on one side L_k and a stationary point such that $A_{\varepsilon,u,z_0}(\rho) < \gamma$

and $\sup_{\Gamma_{\rho/2}(z_0)} \sin^2(u-\alpha_k) \geq \frac{1}{4}$ (notice that by continuity of u this supremum is in fact a maximum). Then there exists $z \in \Gamma_{\rho/2}$ such that $\sin^2(u(z)-\alpha_k) \geq \frac{1}{4}$. We can estimate the $C^{0,1/2}$ seminorm of u on Γ_ρ as in [35, Proposition 3.6], using Sobolev embedding in one dimension as:

$$
\begin{aligned}
[u]^2_{C^{0,1/2}(\Gamma_\rho)} &\leq \int_{\Gamma_\rho} \left|\frac{\partial u}{\partial \tau}\right|^2 \overset{(\dagger)}{\leq} C\left(\frac{A(\rho)}{\rho} + \frac{1}{\varepsilon^2}\int_{\Gamma_\rho} \sin^2(u-\alpha_k)\, d\mathcal{H}^1\right) \\
&\leq \frac{2C\gamma}{\varepsilon},
\end{aligned}
\tag{5.23}
$$

where in the inequality ($\dagger$) we use the inequality (3.9) in [35] (which we can apply since $u \in H^2(\Omega)$ by Lemma 5.3). So we get from this that $\sin^2(u(z')-\alpha_k) \geq \frac{1}{8}$ for all z' such that $|z-z'| \leq \frac{\varepsilon}{C\gamma}$, where the right-hand side is $\geq \frac{\varepsilon}{2}$ if we choose γ small enough. Then we can estimate the penalty term on Γ_ρ as follows:

$$
\frac{1}{2\varepsilon}\int_{\Gamma_\rho(z_0)} \sin^2(u-\alpha_k) \geq \frac{1}{2\varepsilon}\int_{\Gamma_{\varepsilon/2}(z_0)} \sin^2(u-\alpha_k) \geq \frac{1}{2\varepsilon}\cdot\varepsilon\cdot\frac{1}{8} = \frac{1}{16}. \tag{5.24}
$$

On the other hand we have by the assumption $A_{u,\varepsilon,\rho}(\rho) < \gamma$ and Proposition 5.5 that

$$
\frac{1}{2\varepsilon}\int_{\Gamma_\rho(z_0)} \sin^2(u-\alpha_k)\, d\mathcal{H}^1 \leq \gamma. \tag{5.25}
$$

From (5.24) and (5.25) we get that $\gamma > 1/16$. Since we have obtained this for an arbitrary γ, we get a contradiction. This concludes the proof.

□

We can now prove the main result of this subsection:

Proposition 5.8. *Let u_ε be a sequence of stationary points satisfying the logarithmic energy bound $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$. Then the approximate vortex set S_ε can be covered by finitely many balls of radius ε, such that the corresponding balls of radius $\varepsilon/5$ are disjoint. The number of these balls in uniformly bounded in ε.*

Proof. We follow the proof in [35, Proposition 3.9]. Before we delve into the details of the proof, let us present an outline of it: we first will show that we can cover the set S_ε with larger balls, in such a way that the penalty term is bounded there. Then we show that the number of these balls is uniformly bounded in ε. We then turn our attention to balls of radius ε and show that if such a ball contains points in the set S_ε the penalty term there is bounded from below by a constant. We also prove that each such ball is contained in one of the finitely many (larger) balls of our first collection. Therefore also the balls of radius ε that cover S_ε will have to be uniformly bounded in number. We now present the full details:

Step 1 We start by proving that we can cover S_ε with finitely many (larger) balls of radius $5\varepsilon^{5/6}$, whose number is uniformly bounded in ε. We cover the corners $w_i, i = 1, \ldots, 4$ with balls centred there of radius $5\varepsilon^{5/6}$, i.e. $B_{5\varepsilon^{5/6}}(w_i)$. We then cover $S_\varepsilon \setminus \cup_{i=1}^4 B_{5\varepsilon^{5/6}}(w_i)$ via a collection of balls $\bigcup_{x \in S_\varepsilon \setminus \cup_{i=1}^4 B_{5\varepsilon^{5/6}}(w_i)} B_{5\varepsilon^{5/6}}(x)$. Using Vitali's covering lemma we can then find a new cover, which contains the previous cover, which consists of balls $B_{5\varepsilon^{5/6}}(z_j)$, $z_j \in S_\varepsilon \setminus \cup_{i=1}^4 B_{5\varepsilon^{5/6}}(w_i)$, $j \in J_\varepsilon$, such that the balls $B_{\varepsilon^{5/6}}(z_j)$ are disjoint. We choose radii $\alpha_j \in \left[\varepsilon^{6/7}, \varepsilon^{5/6}\right]$ so that the energy in these balls satisfies

$$\frac{84}{\log \frac{1}{\varepsilon}} E_\varepsilon\left(u_\varepsilon; \Omega \cap B_{\varepsilon^{5/6}}(z_j)\right) \geq A_{\varepsilon, u_\varepsilon, z_j}(\alpha_j) \geq \gamma, \tag{5.26}$$

where γ is defined as in Proposition 5.7: we can apply this, since the centres of these balls are further than $5\varepsilon^{5/6}$ away from corners. The existence of radii α_j as above follows from Lemma 5.6. Since these balls are disjoint and combining the lower bound (5.26) with the upper bound $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ on the energy we can conclude that the number of such balls is bounded by

$$|J_\varepsilon| \leq \frac{84M}{\gamma}. \tag{5.27}$$

Now for those z_j's which are further away than $5\varepsilon^{4/5}$ from a corner using again

Lemma 5.6 we pick radii $\rho_j \in \left[5\varepsilon^{5/6}, 5\varepsilon^{4/5}\right]$ such that:

$$A_{\varepsilon,u_\varepsilon,z_j}\left(\rho_j\right) \leq 60M. \tag{5.28}$$

For those (finitely many, uniformly in ε) points z_j who are closer to a corner w_i, we group them in a ball of radius $10\varepsilon^{4/5}$ centred in the corner. We then choose thanks to Lemma 5.6 four radii $\mu_i, i = 1, \ldots, 4 \in \left[10\varepsilon^{4/5}, 10\varepsilon^{3/4}\right]$ such that

$$A_{\varepsilon,u_\varepsilon,w_i}\left(\mu_i\right) \leq 40M. \tag{5.29}$$

Then we have that there exists a constant C such that for all z_k in our collection of points we have that (from the bound (5.7)) and denoting by r_k either ρ_j or μ_i depending on the case:

$$\frac{1}{2\varepsilon}\int_{\Gamma_{r_k}(z_k)} \sin^2\left(u_\varepsilon - g\right) d\mathcal{H}^1 \leq C, \tag{5.30}$$

where we observe that $r_k \gg \varepsilon$. We also denote by $\mathcal{A} := \{w_i, i = 1, \ldots, 4\} \cup \{z_j : j \in J_\varepsilon\}$.

Step 2 Using the previous step, we now prove that we can cover S_ε with finitely many ε-balls, whose number is uniformly bounded in ε. We take four ε-balls $B_\varepsilon\left(w_i\right), i = 1, \ldots, 4$ centred in the corners. Using again Vitali's covering lemma we cover $S_\varepsilon \setminus \cup_{i=1}^4 B_\varepsilon\left(w_i\right)$ with balls $B_\varepsilon\left(p_k\right), p_k \in S_\varepsilon \setminus \cup_{i=1}^4 B_\varepsilon\left(w_i\right)$, $k \in \mathcal{P}_\varepsilon$ such that the balls $B_{\varepsilon/5}\left(p_k\right)$ are disjoint. We can now estimate the Hölder semi-norm of u_ε as follows, for a constant which depends only on Ω and the constant M in the logarithmic estimate $E_\varepsilon\left(u_\varepsilon\right) \leq M \log\frac{1}{\varepsilon}$:

$$\left[u_\varepsilon\right]_{C^{0,1/2}\left(\Gamma_{r_k}\right)} \leq \frac{C}{\sqrt{\varepsilon}}. \tag{5.31}$$

We prove this in a similar way as we did in (5.23). We now show, both for balls centred in a corner and balls that lie entirely one side that:

$$
\begin{aligned}
[u_\varepsilon]^2_{C^{0,1/2}(\Gamma_{r_k})} &\overset{(*)}{\leq} 4\int_{\Gamma_{r_k}} \left|\frac{\partial u_\varepsilon}{\partial \tau}\right|^2 \leq C\left(\frac{A(r_k)}{r_k} + \frac{1}{\varepsilon^2}\int_{\Gamma_{r_k}} \sin^2(u_\varepsilon - \alpha_k)\right) \\
&\overset{(\dagger)}{\leq} \frac{C}{\varepsilon} A(r_k) \overset{(**)}{\leq} \frac{C}{\varepsilon},
\end{aligned}
\tag{5.32}
$$

which gives the desired estimate (5.31). For a ball B_ρ that lies entirely on one side the inequality $(*)$ is easy to prove from $|u_\varepsilon(x) - u_\varepsilon(y)| \leq \int_x^y |\partial u_\varepsilon/\partial\tau| dt$ using Hölder's inequality (in this case the factor 4 is not necessary): this gives $[u_\varepsilon]_{C^{0,1/2}(\Gamma_\rho)} \leq \left(\int_{\Gamma_\rho} |\frac{\partial u_\varepsilon}{\partial \tau}|^2\right)^{1/2}$ and squaring we get $(*)$. Consider now a ball B_ρ in the corner (w.l.o.g. assume the corner is 0) and let $\Gamma_\rho = \Gamma_x \cup \Gamma_y$, where Γ_x, Γ_y are the parts of Γ_ρ that lie on the x and y axes respectively. Notice first that using the boundary condition we have $\int_{\partial\Omega} |\frac{\partial u_\varepsilon}{\partial \nu}|^2 < \infty$. Now from [35, Lemma 3.3], which we can use since $u_\varepsilon \in H^2(\Omega)$ by Lemma 5.3, we get that $\int_{\partial\Omega} |\frac{\partial u_\varepsilon}{\partial \tau}|^2 < \infty$. Arguing as we did in the flat case we obtain an estimate for the $C^{1/2}$ seminorm on Γ_x, Γ_y. Then by an easy direct computation we can show the desired inequality $(*)$: to do this we write $|u_\varepsilon(x,0) - u_\varepsilon(0,y)| \leq |u_\varepsilon(x,0) - u_\varepsilon(0,0)| + |u_\varepsilon(0,y) - u_\varepsilon(0,0)| \leq \left(\int_{\Gamma_x} |\frac{\partial u_\varepsilon}{\partial \tau}|^2\right)^{1/2} \sqrt{|x|} + \left(\int_{\Gamma_y} |\frac{\partial u_\varepsilon}{\partial \tau}|^2\right)^{1/2} \sqrt{|y|}$. We then obtain the conclusion using that $\int_{\Gamma_x} |\frac{\partial u_\varepsilon}{\partial \tau}|^2 \leq \int_{\Gamma_\rho} |\frac{\partial u_\varepsilon}{\partial \tau}|^2$ (and the same for the integral on Γ_y) and using that $\sqrt{|x|} + \sqrt{|y|} \leq 2\sqrt[4]{x^2 + y^2} = 2|(x,0) - (0,y)|^{1/2}$. Observe also that in inequality $(\dagger)$ we have used the bound for the penalty term given by Proposition 5.4 and Proposition 5.5 in a corner or on one side respectively, and the fact that $r_k \gg \varepsilon$ (to get $\frac{1}{r_k} < \frac{1}{\varepsilon}$ in $(\dagger)$). Then we use the bounds for A for our chose radii r_k given by (5.28) and (5.29) to prove inequality $(**)$ and thus we conclude the proof of (5.32).

Since $p_k \in S_\varepsilon$, we have that $\sin^2(u_\varepsilon - g)(p_k) \geq \frac{1}{4}$ and using the Hölder estimate (5.31) we can see using the same argument as we did for the second inequality in (5.24) that there is a constant $c > 0$ such that

$$
\frac{1}{2\varepsilon}\int_{B_{\varepsilon/5}(p_k)} \sin^2(u_\varepsilon - g)\, d\mathcal{H}^1 \geq c > 0. \tag{5.33}
$$

Observe that – by construction – any ε-ball in our collection is contained in the union of the balls of radius r_k which we constructed in Step 1. Indeed if the ball is entirely contained in one of the corner balls of radius μ_k then there's nothing to prove. If it is not, then it is further away than $5\varepsilon^{4/5} \gg 5\varepsilon^{5/6}$ from the corner, and hence it is contained in the union of the balls of radius ρ_j. Thus we have:

$$
\begin{aligned}
c|\mathcal{P}_\varepsilon| &\leq \sum_{k\in\mathcal{P}_\varepsilon} \frac{1}{2\varepsilon}\int_{B_{\varepsilon/5}(p_k)} \sin^2(u_\varepsilon - g)\, d\mathcal{H}^1 \\
&\leq \sum_{z_j\in\mathcal{A}} \frac{1}{2\varepsilon}\int_{\Gamma_{r_j}(z_j)} \sin^2(u_\varepsilon - g)\, d\mathcal{H}^1 \leq C\left(4 + \frac{84M}{\gamma}\right),
\end{aligned}
\tag{5.34}
$$

which concludes the proof. □

From this we can derive the following useful result:

Proposition 5.9. *There is a constant $C > 0$ such that if u_ε is a sequence of critical points of the energy satisfying the logarithmic energy bound $E_\varepsilon(u_\varepsilon) \leq M \log\frac{1}{\varepsilon}$. Then for the oscillation of u_ε there holds:*

$$
\limsup_{\varepsilon\to 0} \operatorname*{osc}_{\overline{\Omega}} u_\varepsilon \leq C. \tag{5.35}
$$

In particular by adding to u_ε a suitable sequence $t_\varepsilon \in 2\pi\mathbb{Z}$ we can assume that the functions u_ε are bounded in $L^\infty(\Omega)$.

Proof. The proof is the same as in [35, Proposition 5.1] (see also [34, Proposition 5.1]) and uses the fact that outside the (finitely many) ε-balls given by Proposition 5.8 the oscillation of u_ε is bounded by the definition of S_ε, since there we have $|\sin(u_\varepsilon - \alpha_k)| < \frac{1}{2}$. Inside the balls $B_\varepsilon(a_i^\varepsilon)$ we can use the Hölder estimate (5.31) – which holds both for balls in the corners and for balls on the flat part as noted there – to get that the oscillation is bounded there as well: we can do this since each ball $B_\varepsilon(a_i^\varepsilon))$ is contained in one of the balls $B_{r_j}(z_j)$ constructed in Step 1 of Proposition 5.8. By the maximum principle then we conclude that the oscillation is bounded on Ω by a constant independent of ε and from this we get (5.35). The second claim then follows immediately. □

We also have the following result, which will be useful later:

Lemma 5.10. *We can cover the set S_ε with a uniformly (in ε) bounded number of balls $B_{\sigma\varepsilon}\left(a_i^\varepsilon\right), i = 1, \ldots, n_0$, for some $\sigma > 0, n_0 \in \mathbb{N}$, where $a_i^\varepsilon \in \partial\Omega$ (which can be chosen such that $a_i^\varepsilon \in S_\varepsilon$), such that these balls are disjoint and for $i \neq j$ we have $|a_i^\varepsilon - a_j^\varepsilon| \gg \varepsilon$.*

Proof. We cover S_ε with balls $B_\varepsilon\left(a_i^\varepsilon\right)$ thanks to Proposition 5.8, so that $a_i^\varepsilon \in S_\varepsilon$. Consider first sequences of points $a_i^\varepsilon, i = 1, \ldots, N$ which lie on one side of the rectangle and that converge to the corner 0 (w.l.o.g. we can assume the corner is in the origin and the points we consider lie on the x axis): let now $C_i \in [0, \infty]$ be defined as

$$C_i := \liminf_{\varepsilon \to 0} \frac{|a_i^\varepsilon|}{\varepsilon}. \tag{5.36}$$

By choosing subsequences finitely many times we can assume that for all i the corresponding sequences converge to the $\liminf$, that is:

$$C_i := \lim_{\varepsilon \to 0} \frac{|a_i^\varepsilon|}{\varepsilon}. \tag{5.37}$$

Define $\mathcal{S} := \{i : C_i < \infty\}$, i.e. the set of indices corresponding to the points which have asymptotically a distance of order ε from the corner and set $C := \max\{C_i : i \in \mathcal{S}\}$. Then for some $\varepsilon_0 > 0$ we can assume that for all $\varepsilon < \varepsilon_0$ and for $i \in \mathcal{S}$ we have

$$B_\varepsilon\left(a_i^\varepsilon\right) \in B_{2(C+1)\varepsilon}\left(0\right), \tag{5.38}$$

so we replace these balls with $B_{2(C+1)\varepsilon}\left(0\right)$. Consider now points a_i^ε such that $i \notin \mathcal{S}$ and the corresponding balls $B_\varepsilon\left(a_i^\varepsilon\right)$. By the same process as above (i.e. considering for each i the limits $\lim_{\varepsilon\to 0} \frac{|a_i^\varepsilon - a_j^\varepsilon|}{\varepsilon}$ for $j \notin \mathcal{S}, j \neq i$) we can group these in new balls with centre in one of the $a_i^\varepsilon, i \notin \mathcal{S}$ and radius $\sigma\varepsilon$ for some $\sigma > 0$ (which does not depend on ε), such that these balls are disjoint and have asymptotic distance much larger than ε to each other. The proof for sequences

a_i^ε which converge to a point which is not a corner is also carried out in the same way. This concludes the proof. $\square$

We introduce the following definition:

Definition 5.11 (Degree of transition). *Let the set S_ε be covered by disjoint balls $B_{\sigma\varepsilon}\left(a_i^\varepsilon\right), i = 1, \ldots, N$ (this is possible thanks to Lemma 5.10), for some fixed $N \in \mathbb{N}$. By definition, on $\partial\Omega \setminus \cup_{i=1}^n B_{\sigma\varepsilon}\left(a_i^\varepsilon\right)$ we have $\sin^2\left(u - \alpha_k\right) < \frac{1}{4}$ on each side L_k. Consider a point a_i^ε which is not a corner and the two connected components Γ_1^i, Γ_2^i of $\partial\Omega \setminus \cup_{s=1}^n B_{\sigma\varepsilon}\left(a_s^\varepsilon\right)$ which lie on the two side of the ball $B_{\sigma\varepsilon}\left(a_i^\varepsilon\right)$: let $K_1^i, K_2^i \in \mathbb{Z}$ be such that $\alpha_k + K_j^i\pi$ is the nearest-point projection onto $\alpha_k + \pi\mathbb{Z}$ of u_ε on $\Gamma_j, j = 1, 2$ (this is unique by Corollary 5.2). Then we call $d_i = K_2^i - K_1^i$ the degree of the transition around the point a_i^ε. For the a corner point we can define the degree of the transition in a similar way (in this case the degree will be in $\frac{1}{2} + \mathbb{Z}$).*

Before we can prove Proposition 5.13 we show that for a sequence of critical points u_ε satisfying the logarithmic bound and for ε small enough, away from the vortices we have that the number k_ε defined in Corollary 5.2 is the same for all functions u_ε (possibly by choosing a subsequence). More precisely we have:

Lemma 5.12. *Let u_ε be a sequence of critical points of the energy E_ε satisfying the logarithmic energy bound $E_\varepsilon\left(u_\varepsilon\right) \leq M \log \frac{1}{\varepsilon}$ and let the set S_ε be covered by finitely many disjoint balls $B_{\sigma\varepsilon}\left(a_i^\varepsilon\right), i = 1, \ldots, N$ thanks to Lemma 5.10. By choosing a subsequence we can assume by compactness of the boundary $\partial\Omega$ that $a_i^\varepsilon \to a_i^0 \in \partial\Omega$ (observe that for $i \neq j$ we can still have $a_i^0 = a_j^0$). Call $a_1, \ldots, a_n$ the distinct limit points. Fix $\rho > 0$ small enough so that the balls $B_\rho\left(a_i\right)$ are disjoint and let $\varepsilon_0 > 0$ be such that for all $\varepsilon < \varepsilon_0$ and for all $i = 1, \ldots, N$ we have $B_{\sigma\varepsilon}\left(a_i^\varepsilon\right) \subset B_\rho\left(a_i^0\right)$. Let $z_0 \in \partial\Omega \setminus \cup_{i=1}^n B_\rho\left(a_i\right)$ and $R > 0$ small enough so that the half-ball $B_{2R}^+\left(z_0\right)$ is contained in Ω and $\Gamma_{2R} \cap \left(\cup_{i=1}^n B_\rho\left(a_i\right)\right) = \emptyset$. Then we have that for ε small enough (say for all $\varepsilon < \varepsilon_1 < \varepsilon_0$) the number k_ε defined in Corollary 5.2 is constant on Γ_R and is the same for all $\varepsilon < \varepsilon_1$, possibly by* *choosing a subsequence. Then we get that it is constant (and independent of ε) on the connected component of $\partial\Omega \setminus \cup_{i=1}^{n} B_\rho(a_i)$ that contains Γ_R.*

Proof. Each u_ε is continuous up to the boundary by Lemma 5.3, which implies that k_ε^x is the same for all $x \in \Gamma_{2R}$, since it is a subset of a connected component of $\partial\Omega \backslash S_\varepsilon$. We can then use the local H^1 and H^2 bounds given in [35, Proposition 5.2 and 5.7] to conclude that the sequence u_ε (possibly by adding a term $\pi t_\varepsilon, t_\varepsilon \in \mathbb{Z}$ so that $\|u_\varepsilon\|_{L^\infty}$ is bounded as in Lemma 5.9) converges weakly in $H^2\left(B_R^+\right)$ to a limit function u_*. We have that u_* is constant a.e. on Γ_R thanks to the convergence of the penalty term given by [35, Proposition 5.8]. In particular we have L^2 convergence on Γ_R, which implies pointwise convergence almost everywhere for a subsequence. Then we see that for such a subsequence k_ε has to be the same for all ε, for ε small enough. The last claim then follows by Corollary 5.2. □

5.1.2 Lower bound for the energy in a corner

We are now ready to show the lower bound in a corner using the aforementioned method by Struwe [51, 52] employed also by Kurzke [34, 35]. Consider a vortex placed in a corner, w.l.o.g. in the origin, and we consider the points $a_1^\varepsilon, \ldots, a_N^\varepsilon$ converging to it as $\varepsilon \to 0$. Let $R > 0$ be such that no other are contained in B_R. By considering ε small enough we can actually assume that all such points are contained in $B_{R/2}$.

Proposition 5.13. *Let u_ε be a sequence of critical points of the energy E_ε satisfying the logarithmic energy bound $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ and let the set S_ε be covered by finitely many disjoint balls $B_{\sigma\varepsilon}(a_i^\varepsilon), i = 1, \ldots, N$ thanks to Lemma 5.10. Let z_0 be a corner point: assume that $R > 0$ is small enough so that for all $\varepsilon < \varepsilon_0$ (for a certain $\varepsilon_0 > 0$) there is no other limit point of the a_i^ε contained in $B_R^+(z_0)$ other than the corner, and that on $\Gamma_R \setminus \Gamma_{R/2}$ the nearest-point projections on each side onto $\alpha_k + \pi\mathbb{Z}$ of the values of u_ε differ by $\frac{\pi}{2} + d\pi$ for all $\varepsilon < \varepsilon_0$ (we can assume this by Lemma 5.12 and because all a_i^ε are eventually in $B_{R/2}^+(z_0)$).*

Then we have the following local lower bound for the energy in a ball of radius R around the corner z_0:

$$E_\varepsilon\left(u_\varepsilon; B_R\right) \geq \frac{\pi}{2} \log \frac{R}{\varepsilon} - C, \tag{5.39}$$

for a constant $C > 0$.

Proof. We can cover the approximate vortex set S_ε with a uniformly bounded number of $\sigma\varepsilon$-balls $B_\varepsilon\left(a_i^\varepsilon\right)$ that are disjoint and whose centres are at an asymptotic distance $\gg \varepsilon$ (i.e. $|a_i^\varepsilon - a_j^\varepsilon| \gg \varepsilon$ for $i \neq j$), thanks to Lemma 5.10. Four of these are in the corners, the rest are on the sides.

Step 1 We first prove that the degree of the transition (see Definition 5.11) around each of the balls on the side is ± 1. We will do this by considering blow-ups of the Euler-Lagrange equation around such points. Consider first points (on the same side) $a_i^\varepsilon \to 0$ (as above we assume w.l.o.g. that the corner is the origin and we are working on the positive x axis): from Lemma 5.10 we have that $|a_i^\varepsilon| \gg \varepsilon$ and $|a_i^\varepsilon - a_j^\varepsilon| \gg \varepsilon$ (w.l.o.g we can assume that these points are $a_i^\varepsilon, i = 1, \ldots, n$ and that $a_i < a_{i+1}$). To apply our blow-up argument we first need to show some gradient bounds: fix $\varepsilon > 0$ and $R > 0$ and consider the ball $B_{4R\varepsilon}\left(a_i^\varepsilon\right)$, such that $|a_i^\varepsilon| > 4R\varepsilon$ (for ε small enough this is always possible, since $|a_i^\varepsilon| \gg \varepsilon$. By blow-up of the equation on scale ε around a_i^ε we get that the blown-up function $w_\varepsilon(x) = u_\varepsilon\left(\varepsilon x + a_i^\varepsilon\right)$ satisfies the equation

$$\begin{cases} \Delta w_\varepsilon = 0 & \text{in } B_{4R}^+(0) \\ \frac{\partial w_\varepsilon}{\partial \nu} = -\frac{1}{2}\sin\left(2\left(w_\varepsilon - \alpha_k\right)\right) & \text{on } \Gamma_{4R}. \end{cases} \tag{5.40}$$

We can then use [12, Lemma 2.3] to obtain a bound on the gradient in B_R^+ which only depends on R and an upper bound for $\|w_\varepsilon\|_\infty$: more precisely we have that

$$\|\nabla w_\varepsilon\|_{L^\infty\left(B_R^+\right)} \leq C, \tag{5.41}$$

where $C > 0$ is a constant which only depends on R and an upper bound $\|w_\varepsilon\|_\infty$. Since the L^∞ norm of w_ε is bounded uniformly in ε (since that of u_ε is thanks to Proposition 5.9), we obtain a uniform gradient bound for the functions w_ε in any ball $B_R(0)$ for any fixed $R > 0$, since we can find $\varepsilon_R > 0$ such that for all $\varepsilon < \varepsilon_R$ we have $|a_i^\varepsilon| > 4R\varepsilon$ for all i. This is true because for we have $\liminf_{\varepsilon \to 0} \frac{|a_i^\varepsilon|}{\varepsilon} = \infty$. Let Γ denote the positive x axis and consider for some $R_0 > 0$ small enough (so that the ball $B_{R_0}(0)$ does not contain any limit points of the centres of the balls covering the bad set other than the origin) the set $\Gamma \cap B_{R_0}(0) \setminus (B_{\sigma\varepsilon}(0) \cup_{i=1}^n B_{\sigma\varepsilon}(a_i^\varepsilon))$. This is a union of intervals which we call Γ_i, for $i = 1, \ldots n$, defined as:

$$\begin{aligned}
&\Gamma_1 := [\sigma\varepsilon, a_1^\varepsilon - \sigma\varepsilon] \\
&\Gamma_i := \left[a_i^\varepsilon + \sigma\varepsilon, a_{i+1}^\varepsilon - \sigma\varepsilon\right] \text{ for } i = 2, \ldots, n-1 \\
&\Gamma_n := [a_n^\varepsilon + \sigma\varepsilon, R_0] .
\end{aligned} \tag{5.42}$$

By definition of S_ε, on each Γ_i we have that $\sin^2(u - \alpha_k) < \frac{1}{4}$ and so for some $0 < \delta < \frac{\pi}{4}$ small enough we have that for all i there exists $K_i \in \mathbb{Z}$ such that $|u_\varepsilon - K_i\pi - \alpha_k| \leq \delta$ on Γ_i. Our goal is now to show that for all $i = 1, \ldots, n$

$$|d_i| = |K_i - K_{i+1}| = 1. \tag{5.43}$$

Define the blow-ups w_ε as above: for every $R > 0$ we then get a sequence of functions defined on B_R^+ which are uniformly bounded in ε and whose gradients are also uniformly bounded in ε, thanks to (5.41). We can then apply the Arzelà-Ascoli theorem to get locally uniform convergence (for a subsequence) to a function w defined in $\mathbb{R}^2_+$. We also get locally weak convergence in H^1 thanks to the boundedness of w_ε and the gradient bound (5.41). The limit function is easily seen to satisfy the equation:

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^2_+ \\ \frac{\partial w}{\partial \nu} = -\frac{1}{2}\sin\left(2\left(w - \alpha_k\right)\right) & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{5.44}$$

The solutions of this equation have been classified by Toland [53], see also [35, Theorem 6.1]. They can either be constant (such that $\sin\left(2\left(w-\alpha_k\right)\right)=0$), periodic, or of the form

$$u\left(x,y\right)=\tilde{\sigma}\arctan\frac{x+a}{y+1}+\pi n+\frac{\pi}{2},$$

for some $n\in\mathbb{Z}, a\in\mathbb{R}$ and a sign $\tilde{\sigma}\in\{-1,+1\}$. The periodic solution can be excluded since it contradicts the fact that we can cover S_ε by – uniformly in ε – finitely many $\sigma\varepsilon$-balls. Since $\sin^2\left(w_\varepsilon-\alpha_k\right)>\frac{1}{4}$ at 0 (and so $w_\varepsilon\left(0\right)$ is bounded away from $\alpha_k+j\pi, j\in\mathbb{Z}$), and we have local uniform convergence, we can exclude that the limit function is constant equal to $\alpha_k+j\pi, j\in\mathbb{Z}$. It remains the possibility that it may be constant equal to $\alpha_k+j\frac{\pi}{2}$: this however cannot be, since we have that $\left|w_\varepsilon\left(2\sigma\right)-\alpha_k-k_\varepsilon\pi\right|\leq\delta$ for some fixed $0<\delta<\frac{\pi}{4}$ for some k_ε, and so at such a point the value of w_ε is at a distance from $\alpha_k+j\frac{\pi}{2}$ which is bounded from below by a positive constant. Hence w_ε cannot converge to a constant. The only possible solution is then one shaped like an arctan function which has limits at $\pm\infty$ that differ by $\pm\pi$. By rescaling back we see that this proves (5.43) for points a_i^ε which are not corner points but converge to a corner point. The proof for points a_i^ε which converge to a point on the sides is done in the same way, so we will not repeat it.

Step 2 For each of the $\sigma\varepsilon$-balls on the sides, we take a symmetric one on the other side, i.e. if we have $B_{\sigma\varepsilon}\left(c_i^\varepsilon,0\right)$ we add $B_{\sigma\varepsilon}\left(0,c_i^\varepsilon\right)$ and viceversa: if the reflected ball intersects a ball which is already on that side (or if it is at a distance of order ε from it) we merge them together: by Lemma 5.10 this can happen at most with one ball since the centres satisfy $|a_i^\varepsilon-a_j^\varepsilon|\gg\varepsilon, i\neq j$. Observe that the balls we obtain in this way are still disjoint and their centres have a distance $\gg\varepsilon$. So our new cover will have the form (where $|b_i^\varepsilon-b_j^\varepsilon|\gg\varepsilon$ for $i\neq j$):

$$S_\varepsilon\cap B_R\left(0,0\right)\subset B_{\sigma\varepsilon}\left(0,0\right)\cup\bigcup_i B_{\sigma\varepsilon}\left(b_i^\varepsilon,0\right)\cup\bigcup_i B_{\sigma\varepsilon}\left(0,b_i^\varepsilon\right).$$

We then get that, being $I_R=B_R\cap\Gamma_x$, we can write $I_R\setminus\bigcup_i B_{\sigma\varepsilon}\left(b_i,0\right)$ as a finite union of intervals $I_m := [\rho_m, R_m]$, $m = 1, \ldots, \tilde{M}$ (the same is true for intervals J_m on Γ_y, where J_m is symmetric to I_m w.r.t $x = y$). We observe that we have $R_{\tilde{M}} = R$, and $\rho_1 = \sigma\varepsilon$. We further observe that there is a constant $K > 0$ such that $\rho_{m+1} \leq K R_m$ (since ρ_{m+1} and R_m differ by $2\sigma\varepsilon$ and $\rho_{m+1}, R_m \gg \varepsilon$). For each m the intervals I_m and J_m are outside of the set S_ε, so the values of u_ε on I_m and J_m are close to respective wells of the corresponding $\sin^2(\cdot - \alpha_k)$ function (more precisely on each interval on the side L_k we have $|u_\varepsilon - \pi j_m^{(k)} - \alpha_k| \leq \delta$ for some $j_m^{(k)} \in \mathbb{Z}$ and $0 < \delta < \pi/4$), that are $\pi/2 + k_m\pi$ apart (more precisely, their unique nearest-point projections on $\alpha_k + \pi\mathbb{Z}$ differ by $\pi/2 + k_m\pi$ for some $k_m \in \mathbb{Z}$). Let the jump in the corner – i.e. the transition around the ball $B_{\sigma\varepsilon}(0,0)$, which we notice is equal to $\pi/2 + k_1\pi$ – be $\pi/2 + \pi d_{corner}$ and let d_j, d_j' be the degrees of the transitions around $\left(b_j^\varepsilon, 0\right)$ and $\left(0, b_j^\varepsilon\right)$ respectively. By our assumption on the values on $\Gamma_R \setminus \Gamma_{R/2}$ we see that $\pi/2 + \pi d = \sum_j \pi\left(d_j + d_j'\right) + (\pi/2 + \pi d_{corner})$, where by Step 1 $d_j, d_j' \in \{\pm 1, 0\}$[1] for all j. This implies that $|d_{corner}| \leq d + N_0$ for an upper bound $N_0 \in \mathbb{N}$ on the number of vortices, which is uniformly bounded in ε by Proposition 5.8. We get that all k_m are bounded uniformly in ε – since we easily see from $d_j, d_j' \in \{\pm 1, 0\}$ that $|k_m - k_{m+1}| \leq 2$, we have $|k_1| = |d_{corner}| \leq d + N_0$ and there are finitely many numbers k_m. Consider now the disjoint quarter-annuli $D_{\rho_m, R_m}(0,0)$ as defined in Lemma 5.1. We can now estimate the energy on $B_R(z_0) \cap \Omega$ from below using Lemma 5.1 (see also Remark 2) and the fact that all D_{ρ_m, R_m} are pairwise disjoint as follows, where

[1] It is ± 1 if the ball already existed on that side, i.e., if it covers a portion of the set S_ε – this ball may have been merged with the symmetric one of a ball on other side but this does not change the degree. If the ball only comes from a reflection from the other side then the blow up converges to a constant, and there is indeed no part of the bad set contained in there and the degree is 0.

all C_m are uniformly bounded from above in ε – as they depend only on k_m:

$$\begin{aligned}
E_\varepsilon\left(u_\varepsilon; B_R\left(0,0\right)\cap\Omega\right) &\geq \sum_{m=1}^{\tilde{M}} E_\varepsilon\left(u_\varepsilon; D_{\rho_m,R_m}\right) \\
&\overset{(\dagger)}{\geq} \sum_{m=1}^{\tilde{M}} \frac{2}{\pi}\left(\frac{\pi}{2}+k_m\pi\right)^2 \log\frac{R_m}{\rho_m} - C_m \\
&\overset{(*)}{\geq} \frac{\pi}{2}\sum_{m=1}^{\tilde{M}} \log\frac{R_m}{\rho_m} - C\tilde{M} \\
&\overset{(**)}{\geq} \frac{\pi}{2}\log\frac{R}{\varepsilon} - \left(\frac{\pi}{2}\log\sigma + C\tilde{M}\right) \geq \frac{\pi}{2}\log\frac{R}{\varepsilon} - \tilde{C},
\end{aligned} \tag{5.45}$$

where in $(\dagger)$ we used[2] Lemma 5.1, in $(*)$ we used that all C_m are bounded from above uniformly in ε and in $(**)$ we used that $\rho_{m+1} \leq KR_m$ for all $m = 1,\ldots,\tilde{M}-1$ This completes the proof. □

Remark 3. We observe that the lower bound in the above result is clearly not optimal in the case of a general critical point which is not a minimizer, but since we only want to show a lower bound on the energy of minimizers (and hence of all other functions) this is enough for our purposes.

5.2 Energy bounds for minimizers

We now focus on minimizers u_ε of the energy in a rectangle, and critical points which have an energy that differs by that of minimizers by a constant. We want to use Proposition 5.13 to get a global lower bound for the energy of minimizers (and hence for all functions) of the form

$$E_\varepsilon\left(u_\varepsilon\right) \geq 2\pi\log\frac{1}{\varepsilon} - C, \tag{5.46}$$

for a constant $C > 0$. We also want to prove that for a sequence of minimizers the set S_ε can be covered by only 4 balls of radius $\sigma\varepsilon$ centred at the corners, for

[2]Observe that by the discussion above the hypotheses of Lemma 5.1 are satisfied, see also Remark 2.

some $\sigma > 0$. We start by observing that we have the following upper bound for minimizers:

Proposition 5.14. *There exists a constant $c = c(\Omega)$ such that for any sequence of minimizers of E_ε we have:*

$$E_\varepsilon(u_\varepsilon) \leq 2\pi \log \frac{1}{\varepsilon} + c. \tag{5.47}$$

Proof. The proof is the same as in [34, Theorem 4.5] and involves constructing appropriate comparison functions for which we can prove the upper bound: of course this upper bound will hold a fortiori for minimizers. The comparison functions are constructed in disjoint balls of radius R near each corner z_i and give local upper bounds of the form

$$E_\varepsilon(u_\varepsilon; B_R(z_i) \cap \Omega) \leq \frac{\pi}{2} \log \frac{R}{\varepsilon} + C. \tag{5.48}$$

These are then combined with a function defined on the rest of the domain with bounded energy. We will not repeat the proof here, we refer the reader to [34] for more details. □

We can now show that for minimizers (as well as for critical points which have the same energy as minimizers up to a constant, i.e. that satisfy $E_\varepsilon(u_\varepsilon) \leq 2\pi \log \frac{1}{\varepsilon} + C$ for a constant $C > 0$) the approximate vortex set can be covered with four balls of radius $\sigma\varepsilon$ (for some $\sigma > 0$) centred in the corners, and that the degree of the transition in a corner is $\pm\frac{1}{2}$. If the approximate vortex set is covered by 4 corner balls the jump at each corner must be $\pm\frac{\pi}{2}$ because of the lower bounds in corners given in Lemma 5.1 (seel also Remark 2) combined with the upper bound given by Proposition 5.14, since a jump of $\pm\frac{\pi}{2}$ already makes up the singular part of the energy. We then only need to show that the approximate vortex set can be covered by only four balls in the corners. We have:

Proposition 5.15. *For a sequence of minimizer (or critical points whose energy is the same as that of minimizers, up to a constant) there exists a constant $\sigma > 0$*

such that the approximate vortex set can be covered by 4 balls centred at the corners with radius $\sigma\varepsilon$, for all $\varepsilon < \varepsilon_0$, for some $\varepsilon_0 > 0$.

Proof. As we showed in Lemma 5.10 we can cover the set S_ε with finitely many (disjoint) $\sigma\varepsilon$ balls $B_{\sigma\varepsilon}\left(a_i^\varepsilon\right)$ for some $\sigma > 0$, such that the points a_i^ε converge to $a_i^0 \in \partial\Omega$ (if necessary by taking a subsequence). Assume by contradiction that the conclusion of the present proposition is not true, i.e. there is a subsequence $\varepsilon_n \to 0$ (in the following we write ε for ε_n for simplicity since we are only working with this subsequence) there is a ball $B_{\sigma\varepsilon}\left(a_{i_0}^\varepsilon\right)$ which contains points in S_ε and such that the distance of $a_{i_0}^\varepsilon$ from all corners is $\gg \varepsilon$. We have that $a_{i_0}^\varepsilon \to a_0 \in \partial\Omega$ and assume first that a_0 is not a corner point: we can combine the lower bound in corners given by Proposition 5.13 with the lower bound for the energy near $a_{i_0}^\varepsilon$ given in [35] at the end of page 13, to get a contradiction, since the lower bounds in the corners already add up to a global lower bound for the energy of the form $2\pi \log \frac{1}{\varepsilon} - C$ (for a constant $C > 0$ only depending on the sequence u_ε) and the energy of u_ε near $a_{i_0}^\varepsilon$ goes to $+\infty$ as $\varepsilon \to 0$. Therefore we can assume in the rest of the proof that the only limit points a_0 are the corners. Let a, b denote the side lengths and let $0 < R < \frac{1}{2}\min\{a, b\}$. Assume w.l.o.g. that the corner we are considering is 0 and all points a_i^ε that converge to 0 are contained in $B_R(0)$. As in the proof of Proposition 5.13 we can make the cover inside this ball symmetric with respect to the line $x = y$, by adding some extra $\sigma\varepsilon$ balls if necessary. Our cover can then be written as follows for some $\tilde{M} \in \mathbb{N}$, and $0 < a_i^\varepsilon < a_{i+1}^\varepsilon$:

$$B_{\sigma\varepsilon}(0) \cup \bigcup_{i=1}^{\tilde{M}} \left(B_{\sigma\varepsilon}\left(a_i^\varepsilon, 0\right) \cup B_{\sigma\varepsilon}\left(0, a_i^\varepsilon\right)\right).$$

By the way we constructed them we have also that for all $i = 1, \ldots, \tilde{M}$

$$\begin{aligned} a_i^\varepsilon &\gg \varepsilon, \\ |a_i^\varepsilon - a_j^\varepsilon| &\gg \varepsilon \text{ for } i \neq j. \end{aligned}$$

Define $a_0 = 0, a_{M+1} = R$ for ease of notation. Now choose radii $s_i^\varepsilon, i = 1, \ldots, \tilde{M}$ satisfying

$$\varepsilon \ll s_i^\varepsilon \ll \frac{1}{3}\min\{a_i^\varepsilon, |a_i^\varepsilon - a_{i-1}^\varepsilon|, |a_i^\varepsilon - a_{i+1}^\varepsilon|\}. \tag{5.49}$$

Define, for all $i = 1, \ldots, \tilde{M}$, $t_i^\varepsilon := \min\{|a_i^\varepsilon - a_{i-1}^\varepsilon|, |a_i^\varepsilon - a_{i+1}^\varepsilon|\}$. Then we have that:

$$B_{t_i^\varepsilon/3}\left(a_i^\varepsilon, 0\right) \cap B_{t_j^\varepsilon/3}\left(a_j^\varepsilon, 0\right) = \emptyset \text{ for } i \neq j,$$
$$B_{s_i^\varepsilon}\left(a_i^\varepsilon, 0\right) \subset B_{t_i^\varepsilon/3}\left(a_i^\varepsilon, 0\right),$$

and the corresponding inclusions are obviously true for the balls with centres $(0, a_i^\varepsilon)$ by symmetry. Consider the quarter-annuli (defined as in Lemma 5.1, see also Figure 5.1) $D_{\sigma\varepsilon, a_1^\varepsilon - s_1^\varepsilon}(0,0)$, $D_{a_{\tilde{M}}^\varepsilon + s_{\tilde{M}}^\varepsilon, R}(0,0)$ and for $i = 1, \tilde{M}-1$ the quarter-annuli $D_{a_i^\varepsilon + s_i^\varepsilon, a_{i+1}^\varepsilon - s_{i+1}^\varepsilon}(0,0)$. Then we can estimate the energy from below on the union of these annuli as follows (in the same way as in Proposition 5.13, since they are disjoint and since there is a constant $K > 0$ such that $a_i^\varepsilon + s_i^\varepsilon \leq K\left(a_i^\varepsilon - s_i^\varepsilon\right)$, where we use that $s_i^\varepsilon \ll a_i^\varepsilon$):

$$E_\varepsilon\left(u_\varepsilon; D_{\sigma\varepsilon, a_1^\varepsilon - s_1^\varepsilon}(0,0) \cup D_{a_{\tilde{M}^\varepsilon} + s_{\tilde{M}^\varepsilon}, R}(0,0) \bigcup_{i=1}^{\tilde{M}-1} D_{a_i^\varepsilon + s_i^\varepsilon, a_{i+1}^\varepsilon - s_{i+1}^\varepsilon}(0,0)\right) \geq \frac{\pi}{2}\log\frac{R}{\varepsilon} - C_1, \tag{5.50}$$

for a constant $C_1 > 0$. We are now ready to show that the set S_ε can be covered just with $\sigma\varepsilon$-ball in the corners. This will mean that we need to show that in fact we do not have any of the balls with centres $(a_i^\varepsilon, 0)$ or $(0, a_i^\varepsilon)$ for $i = 1, \ldots, M$. We have assumed by contradiction that the balls on the sides exist, in particular that we have some point $(a_i^\varepsilon, 0)$ (w.l.o.g we can assume this point to be on the x axis, the proof in the case in which it is on the y axis is identical) such that there exist $K_L, K_R \in \mathbb{Z}$ with[3] $|K_L - K_R| = 1$ so that for all $\sigma\varepsilon < r < s_\varepsilon^i$

[3]That the transition must be ± 1 follows from Step 1 in the proof of Proposition 5.13, since we are on a side of the rectangle.

$$|u_\varepsilon(a_i^\varepsilon - r, 0) - K_L\pi| < \delta,$$
$$|u_\varepsilon(a_i^\varepsilon + r, 0) - K_R\pi| < \delta.$$

We can do this because for such values of r the points we consider lie outside of S_ε. Then we can use the lower bound for the energy in half-annuli given by [34, Proposition 4.17] on the half-annulus $D_{\sigma\varepsilon, s_i^\varepsilon}(a_i^\varepsilon, 0)$ to obtain the lower bound:

$$E_\varepsilon\left(u_\varepsilon; D_{\sigma\varepsilon, s_i^\varepsilon}(a_i^\varepsilon, 0)\right) \geq \pi \log \frac{s_i^\varepsilon}{\varepsilon} - C_2, \tag{5.51}$$

for some constant $C_2 > 0$. Observe that this half-annulus is disjoint from the quarter-annuli considered above. We can then estimate the energy from below as follows

$$\begin{aligned}
&E_\varepsilon\left(u_\varepsilon; D_{\sigma\varepsilon, a_1^\varepsilon - s_1^\varepsilon}(0,0) \cup D_{a_{\tilde{M}}^\varepsilon + s_{\tilde{M}}^\varepsilon, R}(0,0) \cup \bigcup_{i=1}^{\tilde{M}-1} D_{a_i^\varepsilon + s_i^\varepsilon, a_{i+1}^\varepsilon - s_{i+1}^\varepsilon}(0,0) \cup D_{\sigma\varepsilon, s_i^\varepsilon}(a_i^\varepsilon, 0)\right) \\
&\overset{(\dagger)}{=} E_\varepsilon\left(u_\varepsilon; D_{\sigma\varepsilon, a_1^\varepsilon - s_1^\varepsilon}(0,0) \cup D_{a_{\tilde{M}}^\varepsilon + s_{\tilde{M}}^\varepsilon, R}(0,0) \cup \bigcup_{i=1}^{\tilde{M}-1} D_{a_i^\varepsilon + s_i^\varepsilon, a_{i+1}^\varepsilon - s_{i+1}^\varepsilon}(0,0)\right) \\
&+ E_\varepsilon\left(u_\varepsilon; D_{\sigma\varepsilon, s_i^\varepsilon}(a_i^\varepsilon, 0)\right) \\
&\overset{(*)}{\geq} \frac{\pi}{2} \log \frac{R}{\varepsilon} - C_1 + \pi \log \frac{s_i^\varepsilon}{\varepsilon} - C_2 = \frac{\pi}{2} \log \frac{R}{\varepsilon} + \pi \log \frac{s_i^\varepsilon}{\varepsilon} - C,
\end{aligned} \tag{5.52}$$

for $C = C_1 + C_2$, where in $(*)$ we have used (5.50) and (5.51), and where in $(\dagger)$ we have used the fact that the two sets we are splitting over are disjoint. In the other corners we can use the (possibly less accurate) lower bound given by Proposition 5.13 to get that there we can bound the energy from below by $\frac{\pi}{2} \log \frac{R}{\varepsilon} - C$, for a constant $C > 0$ that is fixed once we choose the sequence. Then, adding up the lower bounds in the four corners, we get the global lower bound for the energy:

$$E_\varepsilon(u_\varepsilon) \geq 2\pi \log \frac{1}{\varepsilon} + 2\pi \log R + \pi \log \frac{s_i^\varepsilon}{\varepsilon} - C_1, \tag{5.53}$$

for a constant $C_1 > 0$. Combining the lower bound in (5.53) and the upper bound in (5.47) we obtain that:

$$\pi \log \frac{s_i^\varepsilon}{\varepsilon} \leq C_1 + c(\Omega) - 2\pi \log R < \infty.$$

On the other hand we have that $s_i^\varepsilon \gg \varepsilon$ and so as $\varepsilon \to 0$ we have

$$\pi \log \frac{s_i^\varepsilon}{\varepsilon} \to +\infty,$$

which gives a contradiction. Hence we conclude that we cannot have any balls on the side, and so that S_ε can be covered by 4 balls of radius $\sigma\varepsilon$ centred in the corners. This concludes the proof.

□

From the previous result and from Proposition 5.13 we can in particular conclude that we have the following lower bound for minimizers (and hence for all functions in H^1 on the rectangle):

Theorem 5.16. *Let u_ε be a sequence of minimizers for the energy E_ε on the rectangle Ω. Then there exists a constant $C > 0$ such that:*

$$E(u_\varepsilon) \geq 2\pi \log \frac{1}{\varepsilon} - C. \tag{5.54}$$

Proof. From Proposition 5.15 we can cover the set S_ε with 4 balls of radius $\sigma\varepsilon$ centred in the corners. The degree of a transition in each corner (which is independent of ε) has to be $\pm\frac{1}{2}$ (i.e. the function has a jump of $\pm\frac{\pi}{2}$), otherwise we could use the lower bounds in Lemma 5.1 (see also Remark 2) to get a contradiction. Then the conclusion follows immediately by either adding up the lower bound given by Proposition 5.13 in each corner or alternatively applying Lemma 5.1 for quarter-annuli $D_{\sigma\varepsilon,R}$ for some radius $R > 0$ in each corner and adding up the lower bounds. □

Chapter 6

Energy expansion for limit functions

In this chapter we will present some results on the convergence of minimizers and critical points which have an energy that is the same of minimizers up to a constant: we show that sequences of minimizers (and critical point as above) on a rectangle converge to a limit function which is harmonic in the interior and constant on the sides, where it takes values in $\{0, \frac{\pi}{2}, \pi\}$ and it jumps by $\frac{\pi}{2}$ in the corners. There are essentially 3 different possible configurations (where two are essentially the same, as we will see below). We will examine the energy of these limit configurations and study which of them has the lowest energy, proving rigorously some results that were analysed numerically by Rave and Hubert [24] and whose proof was sketched in the final chapter of [34].

Some of the results that we need have been proved by Kurzke [34] in the final chapter of his PhD thesis. We report them here for the ease of the reader and to provide the necessary prerequisites for the rest of the chapter. Let Ω be a simply connected plane domain whose boundary consists of the union of some C^2 Jordan arcs that meet at points $a_i \in \partial\Omega, i = 1, \ldots, K$, such that at this points there is an exterior angle $\alpha_i \in (-\pi, \pi)$.

Proposition 6.1 ([34], Proposition 8.1)**.** *There exists a minimizer (k_i) to the following problem:*

$$M(\alpha_1, \dots, \alpha_K) := 2\pi - \frac{1}{2}\sum_i \alpha_i + \frac{\pi}{2} \min_{(k_i)\in\mathbb{Z}^N, \sum_i k_i = 2} \sum_i \left(-1 + (k_i - 1)^2 \frac{\pi}{\pi - \alpha_i}\right), \tag{6.1}$$

where we set $\alpha_i = 0$ for $i > K$.

Theorem 6.2 ([34], Theorem 8.2)**.** *For a sequence of minimizers u_ε of E_ε, we have the bound*

$$E_\varepsilon(u_\varepsilon) \le M(\alpha_1, \dots, \alpha_K) \log \frac{1}{\varepsilon} + C, \tag{6.2}$$

for some constant $C > 0$.

We have the following convergence result for sequences of critical points satisfying the same energy bound as minimizers[1]:

Theorem 6.3 ([34], Theorem 8.3)**.** *Let (u_ε) be a sequence of critical points satisfying the energy bound:*

$$E_\varepsilon(u_\varepsilon) \le M(\alpha_1, \dots, \alpha_K) \log \frac{1}{\varepsilon} + C, \tag{6.3}$$

for some $C > 0$. Then there exists a subsequence and finitely many points $p_1, \dots, p_N \in \partial\Omega$ such that for every Ω' with $\overline{\Omega'} \subset \Omega \setminus \{p_1, \dots, p_N\}$ we have:

$$\int_{\Omega'} |\nabla u_\varepsilon|^2 \le C(\Omega'). \tag{6.4}$$

For all $p < 2$ we have the following bound

$$\int_{\Omega} |\nabla u_\varepsilon|^p \le C(p, \Omega). \tag{6.5}$$

[1]The necessary prerequisites have been proved in previous chapters, and the proof is essentially the same as in the case of a smooth boundary, and is sketched in [34]. We will not repeat it here.

In particular, after adding a sequence $z_\varepsilon \in 2\pi\mathbb{Z}$ if necessary we have for a subsequence the convergence $u_\varepsilon \to u_*$ in $H^1(\Omega')$ for Ω' as above and $u_\varepsilon \rightharpoonup u_*$ in $W^{1,p}(\Omega)$ for all $p < 2$. The limit u_* is a harmonic function such that $u - g \in \pi\mathbb{Z}$ is constant on the sides, and jumps at the points p_i. The choice of points and jump height corresponds to the minimization of (6.1).

We conclude this series of results with the following observation:

Remark 4 (Remark 8.5 in [34]). If Ω is a convex domain, the minimizer of (6.1) is given by two vortices in the most acute interior angles. For a rectangle we then have two possibilities: either the two vortices are on the same side (which is called a "C" state) or they are opposite along a diagonal (which is called an "S" state) - see for example [4, Fig. 5.19(a)-(b)] for the C and S states in a rectangle.

In the rest of the chapter we will prove an energy expansion for limit configuration and give a rigorous and quantitative proof that the S state is minimizing: this will involve looking at the second order term in the energy expansion, since both states have the same leading order term.

6.1 Renormalized energy in half-plane

As a preparatory result we compute the following energy expansion in the half-plane for a sum of weighted argument functions around finitely many points on the boundary:

Proposition 6.4. *Let $v(z) := \sum_k d_k \arg(z - b_k)$ be defined*[2] *on $\overline{\mathbb{R}^2_+} \setminus b_i$ for $d_k \in \mathbb{R}$ and points $b_i \in \partial\mathbb{R}^2_+$ and let $\rho_k < \frac{1}{2}\min_{i\neq j}|b_i - b_j|$ be such that $\rho_k \to 0$. Let $R > \max\{|b_i|, \rho_k\}$. Then there exist $\rho_0, R_0, C_1, C_2 > 0$ such that for every $R > R_0, \rho < \rho_0$ we have that*

[2]Here we notice that the argument function is multivalued. We choose the branch which is equal to 0 on $x > 0, y = 0$ and to π on $x < 0, y = 0$, for $z = x + iy \in \mathbb{C}$, with a branch cut in the half-plane $y < 0$.

$$\left|\int_{B_R^+\setminus\bigcup_k B_{\rho_k}(b_k)}|\nabla v|^2-\left(-\pi\sum_k d_k^2\log\rho_k+\pi\left(\sum_k d_k\right)^2\log R+W(b_k,d_k)\right)\right|$$
$$\leq C_1\sum_k\rho_k\log\rho_k+C_2\frac{\log R}{R},$$

where $W(b_k,d_k)=-\pi\sum_{i\neq j}d_jd_i\log|b_i-b_j|$.

Proof. We observe that since v is harmonic, we can consider its harmonic conjugate, which in this case will be the function u defined as:

$$u(z):=\sum_{k=1}d_k\log|z-\beta_k| \tag{6.6}$$

and we observe that by definition of harmonic conjugate the integral of $|\nabla v|^2$ will be equal to that of $|\nabla u|^2$ so we turn to computing the latter. Since u is harmonic we get:

$$\int_{B_R^+\setminus\bigcup_k B_{\rho_k}(\beta_k)}|\nabla u|^2=\int_{\Gamma_R\setminus\bigcup_k B_{\rho_k}^k}u\frac{\partial u}{\partial\nu}+\sum_k\int_{C_{\rho_k}^k}u\frac{\partial u}{\partial\nu}+\int_{C_R}u\frac{\partial u}{\partial\nu}.$$

For the first two terms we can compute the energy expansion as in [34, Proposition 3.12], to get

$$\int_{\Gamma_R\setminus\bigcup_k B_{\rho_k}(\beta_k)}u\frac{\partial u}{\partial\nu}+\sum_k\int_{C_{\rho_k}^k}u\frac{\partial u}{\partial\nu}=-\pi\sum_k d_k^2\log\rho_k+W(\beta_k,d_k)+\sum_k O(\rho_k\log\rho_k)\,. \tag{6.7}$$

The idea of the proof is to write u near each point b_k as the sum of a singular term $\log|z-b_k|$ and a smooth function $S(z)$: then the conclusion follows from a quick computation. Thus we only need to expand the integral on C_R. We first notice that the gradient of u is equal to

$$\nabla u(z)=\sum_k d_k\frac{z-\beta_k}{|z-\beta_k|^2}. \tag{6.8}$$

Then the integral is equal to:

$$\begin{aligned}\int_{C_R} u\frac{\partial u}{\partial \nu} &= \int_{C_R}\left(\sum_{k=1} d_k \log|z-\beta_i|\right)(\nabla u \cdot \nu)\\ &= \int_{C_R}\left(\sum_{k=1} d_k \log|z-\beta_k|\right)\left(\sum_k d_k \frac{z-\beta_k}{|z-\beta_k|^2}\cdot\frac{z}{|z|}\right)\\ &= \int_{C_R}\left(\sum_k d_k \log|z| + \sum_k d_k \log\frac{|z-\beta_k|}{|z|}\right)\\ &\left(\sum_k d_k\frac{1}{|z|} + \sum_k d_k\left(\frac{z-\beta_k}{|z-\beta_k|}\frac{z}{|z|} - \frac{1}{|z|}\right)\right).\end{aligned}$$

Now we have that:

$$\begin{aligned}\frac{z-\beta_i}{|z-\beta_i|^2}\cdot\frac{z}{|z|} - \frac{1}{|z|} &= \frac{(z-\beta_i)\cdot z - |z-\beta_i|^2}{|z||z-\beta_i|^2}\\ &= \frac{(z-\beta_i)\cdot z - (z-\beta_i)\cdot(z-\beta_i)}{|z||z-\beta_i|^2}\\ &= \frac{(z-\beta_i)(z-z+\beta_i)}{|z||z-\beta_i|^2} = \frac{\beta_i\cdot(z-\beta_i)}{|z||z-\beta_i|^2}.\end{aligned} \tag{6.9}$$

So what we need to compute is (using $|z| = R$):

$$\int_{C_R}\left(\sum_k d_k \log R + \sum_k d_k \log\frac{|z-\beta_k|}{|z|}\right)\left(\sum_k d_k\frac{1}{R} + \sum_k d_k\frac{\beta_k\cdot(z-\beta_k)}{|z||z-\beta_k|^2}\right). \tag{6.10}$$

Expanding the product on the right hand side we get:

$$\begin{aligned}\int_{C_R}\left(\sum_k d_k\right)^2\frac{\log R}{R} &+ \int_{C_R}\sum_{k,j} d_k d_j\frac{\beta_j\cdot(z-\beta_j)}{|z||z-\beta_j|^2}\log R\\ &+ \int_{C_R}\sum_{k,j} d_k d_j\frac{1}{R}\log\frac{|z-\beta_j|}{|z|}\\ &+ \int_{C_R}\sum_{k,j} d_k d_j \log\frac{|z-\beta_k|}{|z|}\cdot\frac{\beta_j\cdot(z-\beta_j)}{|z-\beta_j|^2|z|}.\end{aligned} \tag{6.11}$$

Now, observe that all points β_k are contained in a ball of fixed radius C (by Lemma 8.7), so $|\beta_k| \le C$. For large $|z|$ we clearly have that $|z - \beta_k| \sim |z|$. By triangle inequality we have that

$$|z| \le |z - \beta_i| + |\beta_i| \text{ and } |z - \beta_i| \le |z| + |\beta_i|, \tag{6.12}$$

from which we conclude that

$$||z| - |z - \beta_i|| \le |\beta_i|. \tag{6.13}$$

The first integral in (6.11) is equal to:

$$\int_{C_R} \left(\sum_k d_k \right)^2 \frac{\log R}{R} = \pi \left(\sum_k d_k \right)^2 \log R. \tag{6.14}$$

For the second integral we estimate as:

$$\begin{aligned} &\left| \int_{C_R} \sum_{k,j} d_k d_j \frac{\beta_j \cdot (z - \beta_j)}{|z||z - \beta_j|^2} \log R \right| \\ &\le \sum_{k,j} \int_{C_R} \frac{|\beta_j| \cdot |z - \beta_j|}{|z||z - \beta_j|^2} \log R. \end{aligned} \tag{6.15}$$

We now use the trivial estimate $|z - \beta_i| \ge \frac{1}{2} R$ to estimate this from above as

$$\begin{aligned} &\sum_{k,j} \int_{C_R} \frac{|\beta_j| \cdot |z - \beta_j|}{|z||z - \beta_j|^2} \log R \\ &= \sum_{k,j} \int_{C_R} \frac{|\beta_i| \cdot |z - \beta_j|}{|z||z - \beta_j|^2} \log R \\ &\le 2C \sum_{k,j} \int_{C_R} \frac{|\beta_j|}{|z||z - \beta_j|} \log R \\ &\le 2C \sum_{k,j} \int_{C_R} \frac{\log R}{R^2} \le C \frac{\log R}{R}. \end{aligned} \tag{6.16}$$

The third integral in (6.11) can be computed using the Taylor expansion for $\log(1+x)$: we have $\frac{|z-\beta_j|}{|z|} = 1 - \frac{|z|-|z-\beta_i|}{|z|}$ and use (6.13) to get the estimate:

$$\left| \frac{|z| - |z-\beta_i|}{|z|} \right| \leq \frac{C}{R}. \tag{6.17}$$

We can now proceed to estimate the integral as:

$$\begin{aligned} &\int_{C_R} \sum_{k,j} d_k d_j \frac{1}{R} \log \frac{|z-\beta_j|}{|z|} \\ &\qquad \leq \left(\int_{C_R} \sum_{k,j} d_k d_j \frac{1}{R} \left(\frac{C}{R} \right) \right) + O\left(1/R^2\right) \\ &\qquad \leq \frac{\pi}{R} \left(\sum_k d_k \right)^2 + O\left(\frac{1}{R^2} \right) = O\left(\frac{1}{R} \right). \end{aligned} \tag{6.18}$$

The fourth integral can be estimated as:

$$\begin{aligned} &\int_{C_R} \sum_{k,j} d_k d_j \log \frac{|z-\beta_k|}{|z|} \cdot \frac{\beta_j \cdot (z-\beta_j)}{|z-\beta_j|^2 |z|} \\ &\leq \sum_{k,j} d_k d_j \int_{C_R} \left| \log \frac{|z-\beta_k|}{|z|} \right| \cdot \frac{|\beta_j|}{|z||z-\beta_j|}. \end{aligned} \tag{6.19}$$

Since for $|z|$ large enough we have $\frac{1}{2} \leq \frac{|z-\beta_i|}{|z|} \leq \frac{3}{2}$, we can estimate

$$\left| \log \frac{|z-\beta_k|}{|z|} \right| \leq C_1, \tag{6.20}$$

and so obtain the following estimate for the integral, with a constant $C > 0$:

$$\begin{aligned} &\sum_{k,j} d_k d_j \int_{C_R} \left| \log \frac{|z-\beta_k|}{|z|} \right| \cdot \frac{|\beta_j|}{|z||z-\beta_j|} \\ &\qquad \leq \sum_{k,j} d_k d_j C \int_{C_R} \frac{1}{R^2} = O\left(\frac{1}{R} \right). \end{aligned} \tag{6.21}$$

Combining all the estimates in (6.7),(6.14), (6.16), (6.18) and (6.21) we get the conclusion.

□

6.2 Conformal transformation to the half-plane

Consider a rectangle R in $\mathbb{R}^2$ (which in the following we identify with the complex plane $\mathbb{C}$ if convenient) given by

$$R = \{z \in \mathbb{C} : |\Re(z)| < a, 0 < \Im(z) < b\},$$

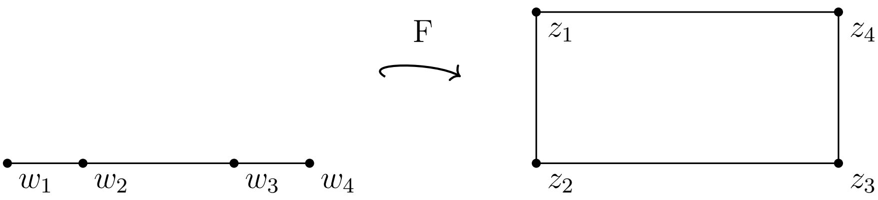
for $a, b > 0$. We denote by z_i, $i = 1, \ldots, 4$ the vertices of the rectangle in the following way: $z_1 = -a + ib, z_2 = -a, z_3 = a, z_4 = a + ib$. We want to find the energy expansion for a harmonic function with boundary value which is constant on each side of the rectangle in the set

$$R_\rho := R \setminus \bigcup_{i=1}^{4} B_\rho(z_i).$$

To find the energy expansion in a rectangle we first conformally transform the rectangle R to the upper half-plane $H := \{z \in \mathbb{C} : \Im(z) > 0\}$. This has the advantage that here we can explicitly solve the boundary problem, and the solution is given by a sum of argument functions. This will allow us to compute the desired energy expansion, using the conformal invariance of the Dirichlet integral. For the transformation we employ the inverse Schwarz-Christoffel map which maps the upper half-plane to the rectangle. Let $0 < k < 1$ (where k depends on the dimensions of the rectangle R) and let $w_i, i = 1, \ldots, 4$ be the points given by

$$w_1 = -\frac{1}{k}, w_2 = -1, w_3 = 1, w_4 = \frac{1}{k}. \tag{6.22}$$

Then the Schwarz-Christoffel map is given by



$$F(w) = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{6.23}$$

This map satisfies $F(w_i) = z_i$ for $i = 1, \ldots, 4$ and maps the upper half-plane H to the interior of the rectangle. The map is conformal in H and on the boundary away from the points w_i. The integral in (6.23) is known as the *incomplete elliptic integral of the first kind*. The inverse function of F is *Jacobi's elliptic function* sn, also known as *elliptic sine* or *sinus amplitudinis* from its Latin name. For an introduction to elliptic function we refer to Chapter VI of Zeev Nehari's book [47] or Neville's book [48]. A useful list of identities and properties can also be found on the website of the NIST Digital Library of Mathematical Functions (`https://dlmf.nist.gov/`)

To be able to compute the energy in the half-plane we then need to understand how the balls around the vertices z_i transform through sn. Let $\rho > 0$ be small and consider $\Gamma_\rho^i := \partial B_\rho(z_i) \cap R$. We want to study how Γ_ρ is transformed under sn, so that we can understand how the image of R_ρ looks like. To do this we first find a Taylor expansion for $\operatorname{sn} z$ in a neighbourhood of each point z_i: in the corners, we have $\operatorname{sn}'(z_i) = 0$: (see `https://dlmf.nist.gov/22.5` for the values of the derivative of sn at the corners of the rectangle). We can also use the following identities to get the values of the derivatives, using the fact that $\operatorname{cn}(K) = 0 = \operatorname{dn}(K + iK')$:

$$\frac{\operatorname{sn}(z)}{dz} = \operatorname{cn}(z)\operatorname{dn}(z). \tag{6.24}$$

Here cn and dn are the Jacobi *elliptic cosine* (or *cosinus amplitudinis*) and *delta amplitude* (or *delta amplitudinis*) respectively.

Hence, near a point z_i, we can express the function sn using its Taylor expansion as:

$$\operatorname{sn}(z) = \operatorname{sn}(z_i) + \frac{1}{2}\frac{d^2 \operatorname{sn}}{dz^2}(z_i)(z-z_i)^2 + O(z-z_i)^3 . \tag{6.25}$$

To find the values of the second derivative we use a second-order differential equation which is satisfied by $y = \operatorname{sn} z$ (see https://dlmf.nist.gov/22.13#iii), namely

$$\frac{d^2 y}{dz^2} = -\left(1+k^2\right) y + 2k^2 y^3. \tag{6.26}$$

Then a simple calculation shows that

$$\begin{aligned} \frac{d^2 y}{dz^2}(a) &= k^2 - 1 < 0 \\ \frac{d^2 y}{dz^2}(-a) &= 1 - k^2 > 0 \\ \frac{d^2 y}{dz^2}(a+ib) &= \frac{1}{k} - k > 0 \\ \frac{d^2 y}{dz^2}(-a+ib) &= k - \frac{1}{k} < 0. \end{aligned} \tag{6.27}$$

From this we have that in the Taylor expansion in (6.25) the second derivative is never zero for the points we consider. Therefore with $C_i := \left|\frac{1}{2}\frac{d^2 \operatorname{sn}}{dz^2}(z_i)\right|$ we find that for a suitable constant $c_i > 0$ and for $\rho > 0$ small enough

$$\operatorname{sn}\left(\Gamma_\rho^i\right) \subset \left(B_{C_i\rho^2 + c_i\rho^3} \setminus B_{C_i\rho^2 - c_i\rho^3}\right) \cap H. \tag{6.28}$$

So can estimate the energy on $\operatorname{sn}(R_\rho)$ of $\tilde{u} = u \circ F$ as follows:

$$E\left(\tilde{u}, H \setminus \bigcup_{i=1}^{4} B_{C_i\rho^2 + c_i\rho^3}(w_i)\right) \le E\left(\tilde{u}, \operatorname{sn}(R_\rho)\right) \le E\left(\tilde{u}, H \setminus \bigcup_{i=1}^{4} B_{C_i\rho^2 - c_i\rho^3}(w_i)\right).$$

We can now compute explicitly the two energies on the sides to get an estimate of the energy in between. We recall Theorem 6.4. In our case we have $\sum_k d_k = 0$, so taking the limit for $R \to \infty$ we can conclude that for every $r > 0$ small enough we have:

$$E\left(\tilde{u}, H \setminus \bigcup_{k=1}^{4} B_r(w_i)\right) = -\pi \sum_{k=1}^{4} d_k^2 \log r + W(w_k, d_k) + O(r \log r). \tag{6.29}$$

We now use this to compute the energies $E\left(\tilde{u}, H \setminus \bigcup_{i=1}^{4} B_{C_i\rho^2 + c_i\rho^3}(w_i)\right)$ and $E\left(\tilde{u}, H \setminus \bigcup_{i=1}^{4} B_{C_i\rho^2 - c_i\rho^3}(w_i)\right)$.

We apply Proposition 6.4 with radii $\rho_k = C_k\rho^2 \pm c_k\rho^3$ and obtain

$$\begin{aligned} &E\left(\tilde{u}, H \setminus \bigcup_{i=k}^{4} B_{C_k\rho^2 \pm c_k\rho^3}(w_k)\right) \\ &= -\pi \sum_{k} d_k^2 \log\left(C_k\rho^2 \pm c_k\rho^3\right) - \pi \sum_{i \neq j} d_i d_j \log\left|\operatorname{sn}(z_i) - \operatorname{sn}(z_j)\right| + o(\rho) \\ &= -2\pi \sum_{k} d_k^2 \log \rho - \pi \sum_{k} d_k^2 \log\left(\frac{1}{2}\left|\frac{d^2 \operatorname{sn}}{dz^2}(z_k)\right|\right) + o(\rho) \\ &\quad - \pi \sum_{i \neq j} d_i d_j \log\left|\operatorname{sn}(z_i) - \operatorname{sn}(z_j)\right| + o(\rho). \end{aligned} \tag{6.30}$$

This allows us to conclude that

$$\begin{aligned} E(\tilde{u}, \operatorname{sn}(R_\rho)) = 2\pi \sum_{k=1}^{4} d_k^2 \log \frac{1}{\rho} - \pi \sum_{k=1}^{4} d_k^2 \log\left(\frac{1}{2}\left|\frac{d^2 \operatorname{sn}}{dz^2}(z_k)\right|\right) \\ -\pi \sum_{i \neq j}^{4} d_i d_j \log\left|\operatorname{sn}(z_i) - \operatorname{sn}(z_j)\right| + o(\rho) \end{aligned} \tag{6.31}$$

and by the conformal invariance of the Dirichlet integral we can finally conclude that

$$
\begin{aligned}
E\left(u, R_{\rho}\right)=2 \pi \sum_{k=1}^{4} d_{k}^{2} \log \frac{1}{\rho}-\pi \sum_{k=1}^{4} d_{k}^{2} \log \left(\frac{1}{2}\left|\frac{d^{2} \operatorname{sn}}{d z^{2}}\left(z_{k}\right)\right|\right) \\
-\pi \sum_{i \neq j}^{4} d_{i} d_{j} \log \left|\operatorname{sn}\left(z_{i}\right)-\operatorname{sn}\left(z_{j}\right)\right|+o(\rho),
\end{aligned}
\tag{6.32}
$$

so that the renormalized energy for the rectangle is

$$
W\left(z_{k}, d_{k}\right)=-\pi \sum_{i \neq j}^{4} d_{i} d_{j} \log \left|\operatorname{sn}\left(z_{i}\right)-\operatorname{sn}\left(z_{j}\right)\right|-\pi \sum_{k=1}^{4} d_{k}^{2} \log \left(\frac{1}{2}\left|\frac{d^{2} \operatorname{sn}}{d z^{2}}\left(z_{k}\right)\right|\right).
$$

We can summarize this in the following way:

Proposition 6.5. *Let R be the rectangle*

$$
R=\{z \in \mathbb{C}:|\Re(z)|<a, 0<\Im(z)<b\},
\tag{6.33}
$$

and denote by $z_i, i=1,\ldots,4$ be its vertices. Let u be a harmonic function that has constant values on each side and jumps by $d_i\pi = \frac{s_i}{2}\pi$ at z_i - where $s_i \in \mathbb{Z}$. Let $0<\rho<\min\{a, \frac{b}{2}\}$ and let $R_\rho := R \setminus \cup_{i=1}^{4} B_\rho(z_i)$. Then the energy of u on R_ρ has the following expansion:

$$
\begin{aligned}
E\left(u, R_{\rho}\right) & =2 \pi \sum_{k=1}^{4} d_{k}^{2} \log \frac{1}{\rho}+W\left(z_{k}, d_{k}\right)+o(\rho) \\
& =\frac{\pi}{2} \sum_{k=1}^{4} s_{k}^{2} \log \frac{1}{\rho}+\hat{W}\left(z_{k}, s_{k}\right)+o(\rho)
\end{aligned}
\tag{6.34}
$$

where

$$
\hat{W}\left(z_{k}, s_{k}\right)=-\frac{\pi}{2} \sum_{i<j}^{4} s_{i} s_{j} \log \left|\operatorname{sn}\left(z_{i}\right)-\operatorname{sn}\left(z_{j}\right)\right|-\frac{\pi}{4} \sum_{k=1}^{4} s_{k}^{2} \log \left(\frac{1}{2}\left|\frac{d^{2} \operatorname{sn}}{d z^{2}}\left(z_{k}\right)\right|\right).
\tag{6.35}
$$

6.3 Energy comparison for different configurations

Using the results of the previous section we can now compare the energies of the different configurations (corresponding to the possible boundary values for the limit) in which the jump between the values on adjacent sides is $\pm\frac{\pi}{2}$. Modulo symmetries there are three possibilities, which are the following:

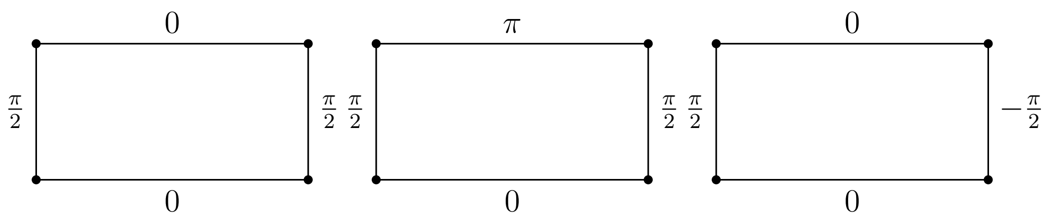


Figure 6.1: Possible configurations

We observe that the second and third configuration are essentially the same (they have the same value on two opposite sides, and values differing by π on the other two), while the first is essentially different. We say that the first is the *S state*, while the other two are example of the *C state*: the difference is that they correspond to different shapes of the rectangles (depending on the value of $k \in (0,1)$), where the pair of sides with the same boundary value is on the short or the long side respectively - in a square they are exactly the same. We have that these configurations correspond to the following choices for s_i:

s_1	s_2	s_3	s_4
1	-1	1	-1
-1	-1	1	1
1	-1	-1	1

We can now use Proposition 6.5 to compute the energy for each of these configurations. The leading term in the energy is the same, so the comparison will involve only the renormalized energy. We can also see that the second part in the renormalized energy $\hat{W}$ in (6.35) is equal to

$$-\frac{\pi}{4}\sum_{i=1}^{4} s_i^2 \log\left(\frac{1}{2}\left|\frac{d^2\,\mathrm{sn}}{dz^2}(z_i)\right|\right) = \pi\log 2 - \log\left(1-k^2\right) + \frac{\pi}{2}\log k,$$

for all three configurations. Thus, in order to compare them, it is enough to consider the first part of $\hat{W}$. This can be written by a direct calculation and using that the values of sn in the corners are known:

$$\begin{aligned}
&-\frac{\pi}{2}\sum_{i<j} s_i s_j \log|\mathrm{sn}(z_i) - \mathrm{sn}(z_j)| \\
&\quad = -\frac{\pi}{2}\Big[\log(1-k)(s_1s_2+s_3s_4) + \log(1+k)(s_1s_3+s_2s_4) \\
&\qquad + \log 2\,(s_1s_4+s_2s_3) - \log k\,(s_1s_2+s_3s_4+s_1s_3+s_2s_4+s_1s_4)\Big].
\end{aligned} \tag{6.36}$$

For the three configurations we can now compute the energy

1. $$\frac{\pi}{2}\Big[2\log(1-k) - 2\log(1+k) + 2\log 2 - \log k\Big]. \tag{6.37}$$

2. $$\frac{\pi}{2}\Big[-2\log(1-k) + 2\log(1+k) + 2\log 2 - \log k\Big]. \tag{6.38}$$

3. $$\frac{\pi}{2}\Big[2\log(1-k) + 2\log(1+k) - 2\log 2 - 3\log k\Big]. \tag{6.39}$$

Subtracting (6.38) from (6.37) we obtain that the difference is equal to

$$\frac{\pi}{2}\Big[4\log(1-k) - 4\log(1+k)\Big] = 2\pi\log\frac{1-k}{1+k} < 0, \tag{6.40}$$

since $\frac{1-k}{1+k} \in (0,1)$. This shows that (6.37) $<$ (6.38), that is the first configuration is energetically favourable compared to the second one.

In the same way we compare the first and third configurations and obtain that the difference of the renormalized energies (6.37)-(6.39) is equal to

$$\frac{\pi}{2}\Big[-4\log(1+k) + 4\log 2 + 2\log k \Big] = 2\pi \log \frac{2\sqrt{k}}{1+k} < 0, \tag{6.41}$$

where we use that $2\sqrt{k} < 1+k$ for $k > 0$. This shows that the first configuration is energetically favourable even compared to the third one. Thus we have that the S state is minimizing, as was hinted in the last chapter of [34]. We can summarize this in the following graph:

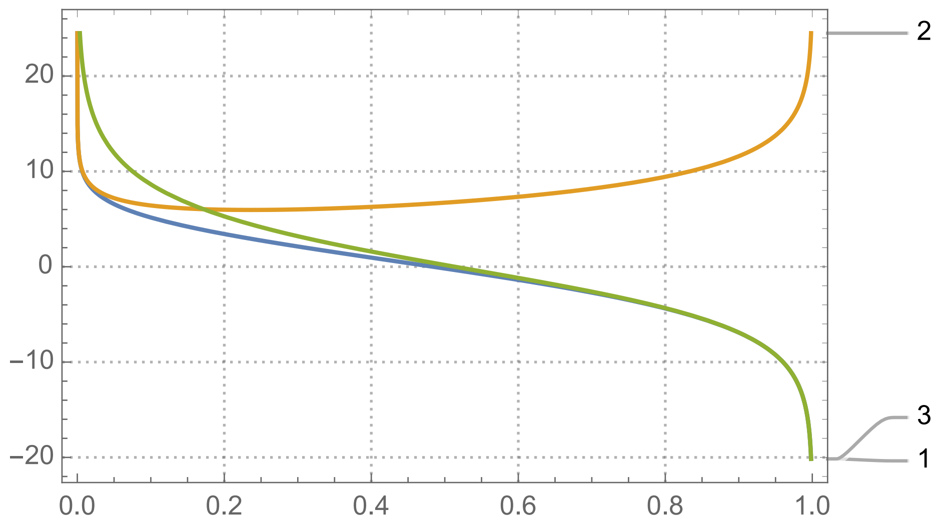


Figure 6.2: Comparison of the energy of the three configurations as a function of the *elliptic modulus* k. The intersection point between the yellow and the green line correspond to a square, in which those two configurations are exactly the same.

In the following graph we compare the first and second configuration:

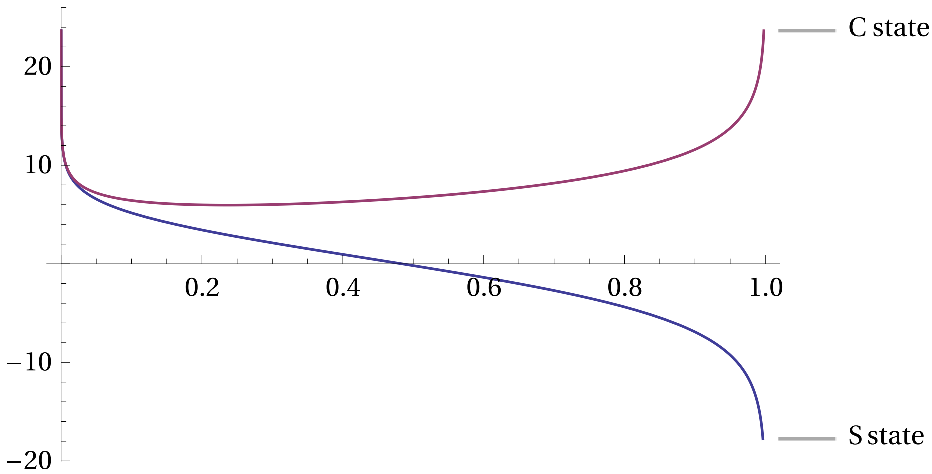
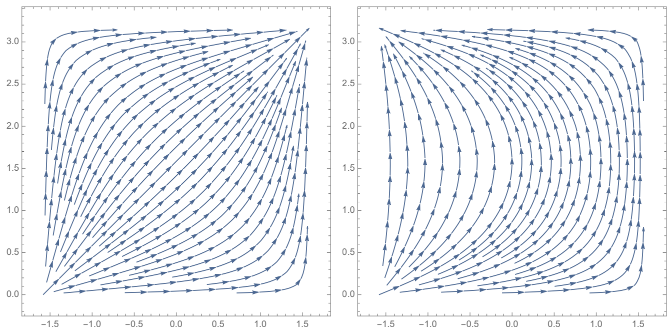


Figure 6.3: Comparison of the energy of the C and S states as a function of the *elliptic modulus* k.

As we have seen, composing a solution in the upper-half plane with the elliptic function $\mathrm{sn}\,(z,k)$ we obtain a solution of the boundary value problem in the rectangle R. Since u is linked to the magnetization M as $M = (\cos u, \sin u)$ we can plot the magnetization in a rectangle with different boundary conditions, depending on the dimensions of the rectangle, which we parametrize by $\tau = \frac{b}{a}$.

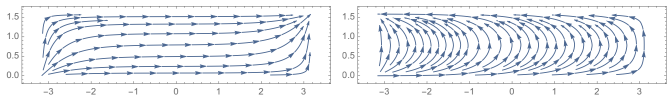
We plot the streamlines for the vector-field M for different values of τ. We start with $\tau = 2$, which corresponds to a square:



(a) (b)

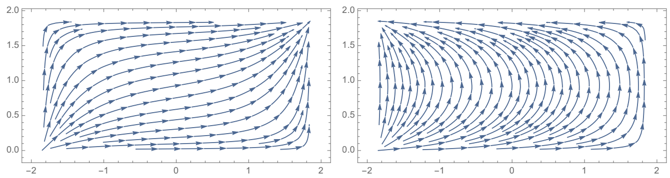
Figure 6.4: (a) The S state in a square. (b) The C state in a square.

We now plot the streamlines for some other values of τ:



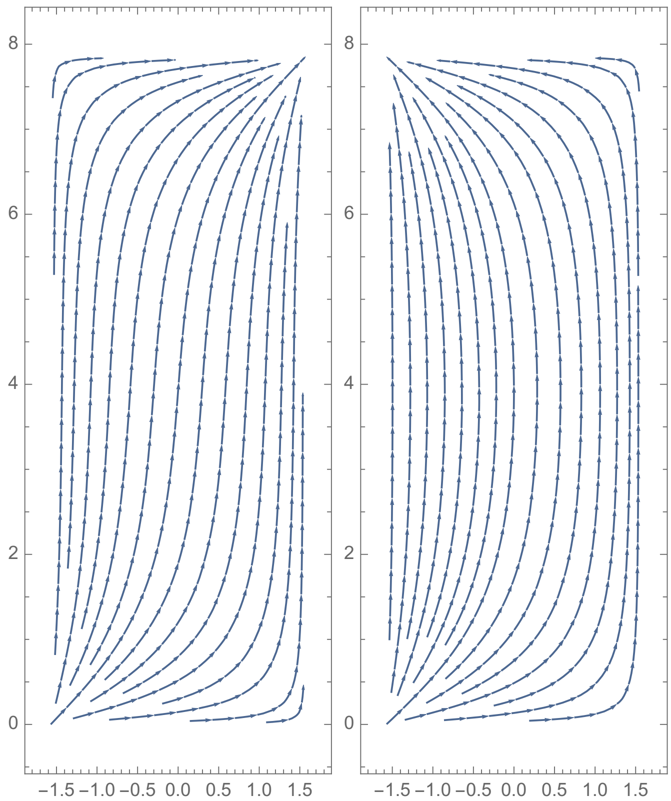
(a) (b)

Figure 6.5: (a) The S state for $\tau = 0.5$. (b) The C state for $\tau = 0.5$.



(a) (b)

Figure 6.6: (a) The S state for $\tau = 1$. (b) The C state for $\tau = 1$.



(a) (b)

Figure 6.7: (a) The S state for $\tau = 5$. (b) The C state for $\tau = 5$.

Chapter 7

Second order lower bounds for the energy

7.1 Second order lower bounds for scalar functionals

In this section we are going to prove second order lower bounds for the energy. We start with the following lemma – which will be relevant later – that establishes the existence of two limits, shows that they are equal and computes the value of the limit:

Lemma 7.1. *Let $\varphi^* (x, y) := \arg(x + iy)$ and $\varphi^*_\varepsilon (x, y) = \arg (x + iy + \varepsilon (1 + i))$ for $(x, y) \in Q$. Setting $\Gamma_r := \{(t, 0) : t \in [0, r)\} \cup \{(0, t) : t \in [0, r)\} = \partial B_r^+ \setminus \partial B_r$ for $r > 0$, we define*

$$\gamma_1 := \liminf_{\varepsilon \to 0} \left(\inf_{\psi = \varphi^* \ \text{on } \partial B_r^+ \setminus \Gamma_r} E_\varepsilon \left(\psi; B_r^+\right) - \frac{\pi}{2} \log \frac{r}{\varepsilon} \right) \tag{7.1}$$

and

$$\gamma_2 := \lim_{r \to 0} \liminf_{\varepsilon \to 0} \left(\inf_{\psi = \varphi^*_\varepsilon \ \text{on } \partial B_r^+ \setminus \Gamma_r} E_\varepsilon \left(\psi; B_r^+\right) - \frac{\pi}{2} \log \frac{r}{\varepsilon} \right). \tag{7.2}$$

Then $\gamma_1 = \gamma_2 = \gamma_0 := \frac{\pi}{2}\log 2e + G = \frac{\pi}{2}(\log 2 + 1) + G$, where G is Catalan's constant defined in Chapter 4.

Proof. Before we start the proof we remark that in the definition of γ_1 it is possible to scale out r by replacing r by 1 and ε by ε/r without changing the result, so in fact the limit does not depend on r. This is because by linear scaling the energy scales in the right way and the boundary condition in unchanged, since φ^* is constant on radial directions.

Step 1 We show that $\gamma_1 = \gamma_2$. In order to do that we construct comparison functions φ_ε on the annulus $B^+_{r(1+r)} \setminus B^+_r$ for some $r > 0$ that equal φ^*_ε on $\partial B^+_{r(1+r)} \setminus \Gamma_{r(1+r)}$ and equal φ^* on $\partial B^+_r \setminus \Gamma_r$. For example we can choose as interpolation functions the following

$$\varphi_\varepsilon(x,y) := \arg\left(x + iy + (1+i)\,\varepsilon\frac{\sqrt{x^2+y^2} - r}{r^2}\right), \tag{7.3}$$

for $(x,y) \in B^+_{r(1+r)\setminus B_r}$.

We can now verify that the energy of these comparison functions is small on the annulus. More precisely we verify that (by dominated convergence) we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{B^+_{r(1+r)\setminus B_r}} |\nabla\varphi_\varepsilon|^2 dxdy &= \int_{B^+_{r(1+r)\setminus B_r}} |\nabla\varphi^*|^2 dxdy \\ &= \int_0^{\frac{\pi}{2}}\int_r^{r(1+r)} \frac{1}{s} ds d\theta = \frac{\pi}{2}\log(1+r),\end{aligned}$$

which tends to 0 as $r \to 0$. Furthermore for the penalty term we have for $t \in (r, r(1+r))$:

$$\sin^2\varphi_\varepsilon(t,0) \le \sin^2\varphi_\varepsilon(r(1+r),0) \le C\left(\frac{\varepsilon}{r}\right)^2.$$

The same estimate is seen to hold for $\sin^2\left(\varphi_\varepsilon(0,t) - \frac{\pi}{2}\right)$ by using the identity $\sin^2\left(\varphi_\varepsilon(0,t) - \frac{\pi}{2}\right) = \sin^2\varphi_\varepsilon(t,0)$, which follows from the fact that $\varphi_\varepsilon(y,x) =$

$\pi/2 - \varphi_\varepsilon(x, y)$. This last thing can be seen by a simple calculation.

Let's now show that the estimate is indeed true. Let us first notice that the argument of $\sin^2$ only takes values between 0 and $\pi/2$, and so we can use the fact that $\sin^2$ is increasing on that interval.

We have from (7.3) that:

$$\varphi_\varepsilon(t, 0) = \arg\left(t + (1+i)\,\varepsilon\frac{t-r}{r^2}\right) = \arg\left(1 + (1+i)\,\varepsilon\frac{t-r}{tr^2}\right),$$

since arg is invariant by radial rescaling. To find an upper bound on the quantity $\sin^2\varphi_\varepsilon(t, 0)$ for $t \in (r, r(1+r))$ we then just need to find an upper bound for the argument. We are looking at complex numbers on a line starting at 1 and at an angle $\pi/4$ from the positive x axis. We can see that the further along we go on that line the greater the argument becomes, and it can be seen easily that the maximum is taken when $t = r(r+1)$, since the derivative of $\frac{t-r}{tr^2}$ in t is positive. So we get (using in (†) again the invariance by radial rescaling)

$$\arg\left(1 + (1+i)\,\varepsilon\frac{t-r}{tr^2}\right) \overset{(\dagger)}{\leq} \arg\left(r(1+r) + (1+i)\,\varepsilon\right) = \arg\left(r^2 + r + \varepsilon + i\varepsilon\right).$$

Now we observe that in the first quadrant (where we are) the argument function is decreasing in the x variable, so we can make it larger by keeping y fixed and decreasing x which for our case means that we have

$$\arg\left(r^2 + r + \varepsilon + i\varepsilon\right) \leq \arg(r + i\varepsilon) = \arctan(\frac{\varepsilon}{r}).$$

Now using the identity $\sin^2(\arctan(x)) = \frac{x^2}{1+x^2}$ and putting everything together we can conclude that

$$\sin^2\varphi_\varepsilon(t, 0) \leq \sin^2\left(\arctan\left(\frac{\varepsilon}{r}\right)\right) \leq \frac{\left(\frac{\varepsilon}{r}\right)^2}{1 + \left(\frac{\varepsilon}{r}\right)^2} \leq \left(\frac{\varepsilon}{r}\right)^2.$$

Consider now an arbitrary function φ in B_r^+ which is equal to φ^* on $\partial B_r^+ \setminus \Gamma_r$ and extend it to a function φ_ε defined on $B_{r(1+r)}^+$ as described in (7.3), so that $\varphi_\varepsilon = \varphi_\varepsilon^*$ on $\partial B_{r(r+1)}^+ \setminus \Gamma_{r(1+r)}$.

$$E_\varepsilon\left(\varphi_\varepsilon; B_{r(r+1)}^+\right) = E_\varepsilon\left(\varphi_\varepsilon; B_r^+\right) + E_\varepsilon\left(\varphi_\varepsilon; B_{r(r+1)}^+ \setminus B_r^+\right).$$

As we have shown above the term $E_\varepsilon\left(\varphi_\varepsilon; B_{r(r+1)}^+ \setminus B_r^+\right)$ tends to 0 in the limit for $\varepsilon \to 0$ and $r \to 0$ (in this order). We have (in the equality (†) we use that $\varphi_\varepsilon = \varphi$ on B_r^+):

$$\begin{aligned} E_\varepsilon\left(\varphi_\varepsilon; B_{r(r+1)}^+\right) - \frac{\pi}{2}\log\frac{r(1+r)}{\varepsilon} =& E_\varepsilon\left(\varphi_\varepsilon; B_r^+\right) - \frac{\pi}{2}\log\frac{r}{\varepsilon} \\ &+ E_\varepsilon\left(\varphi_\varepsilon; B_{r(r+1)}^+ \setminus B_r^+\right) - \frac{\pi}{2}\log(1+r) \\ \stackrel{(\dagger)}{=}& E_\varepsilon\left(\varphi; B_r^+\right) - \frac{\pi}{2}\log\frac{r}{\varepsilon} \\ &+ E_\varepsilon\left(\varphi_\varepsilon; B_{r(r+1)}^+ \setminus B_r^+\right) - \frac{\pi}{2}\log(1+r) \end{aligned}$$

We now take the infimum on all φ such that $\varphi = \varphi^*$ on $\partial B_r^+ \setminus \Gamma_r$ on both sides and we get:

$$\begin{aligned} &\inf_{\psi=\varphi_\varepsilon^* \text{ on } \partial B_{r(r+1)}^+ \setminus \Gamma_{r(1+r)}} E_\varepsilon\left(\psi; B_{r(r+1)}^+\right) - \frac{\pi}{2}\log\frac{r(1+r)}{\varepsilon} \\ \stackrel{(\dagger)}{\leq}& \inf_{\varphi_\varepsilon \text{ defined as in (7.3) from } \varphi=\varphi^* \text{ on } \partial B_r^+ \setminus \Gamma_r} E_\varepsilon\left(\varphi_\varepsilon; B_{r(r+1)}^+\right) - \frac{\pi}{2}\log\frac{r(1+r)}{\varepsilon} \\ \stackrel{(*)}{\leq}& \inf_{\varphi=\varphi^* \text{ on } \partial B_r^+ \setminus \Gamma_r} E_\varepsilon\left(\varphi; B_r^+\right) - \frac{\pi}{2}\log\frac{r}{\varepsilon} + E_\varepsilon\left(\varphi_\varepsilon; B_{r(r+1)}^+ \setminus B_r^+\right) - \frac{\pi}{2}\log(1+r), \end{aligned}$$

where in (†) we use the fact that for φ defined as in (7.3) we have that $\{\varphi_\varepsilon : \varphi_\varepsilon \text{ defined as in (7.3) from } \varphi = \varphi^* \text{ on } \partial B_r^+ \setminus \Gamma_r\} \subset \{\psi = \varphi_\varepsilon^* \text{ on } \partial B_{r(r+1)}^+ \setminus \Gamma_{r(1+r)}\}$ and thus the infimum is larger on the smaller set, and in $(*)$ we use that there φ_ε is univocally defined from φ, and so we can replace the infimum over φ_ε by that over $\varphi, \varphi = \varphi^*$ on $\partial B_r^+ \setminus \Gamma_r$.

Now letting first $\varepsilon \to 0$ and then $r \to 0$ we get that $\gamma_2 \leq \gamma_1$. The opposite inequality follows from a similar interpolation argument.

Step 2 We compute the value of γ. To do this we will use the uniqueness result in Chapter 4. We observe that near a corner we can replace any function with one that does not increase the energy and lies between 0 and $\pi/2$: we have already observed on page 62 that we can replace any function with one that lies between 0 and π and that does not increase the energy. Near a corner we consider the set $A = \{\pi/2 < u \leq \pi\} \cap B_r^+$: on such set we redefine u as $\pi - u$: this gives us a function which is still in H^1 and for which the Dirichlet energy does not increase. Furthermore we notice that $\sin^2(\pi - u - \pi/2) = \sin^2(u - \pi/2)$ and in the same way $\sin^2(\pi - u) = \sin^2(u)$, so the penalty terms are also left unchanged. Finally, we observe that if we start from a function that is equal to φ_ε^* on $\partial B_r^+ \setminus \Gamma_r$ this condition will still be true, since we do not change the function on that set (since $0 < \varphi_\varepsilon^* < \pi/2$ there). In particular if we start from a minimizer, we obtain a minimizer, call it v, that lies between 0 and $\pi/2$. Then we can use the uniqueness of a solution with the given boundary conditions and bounds 0 and $\pi/2$ proved in Chapter 4 to see that this minimizer has to be φ_ε^*: as a minimizer it has to satisfy the Euler-Lagrange equation with the given boundary condition. If we rescale this as $\psi(z) := v(\varepsilon z)$ we have that ψ is a solution of (4.8) for a radius r/ε – the Dirichlet condition on $\partial B_{r/\varepsilon} \cap Q$ is satisfied since u is obtained rescaling φ_ε^* in the same way and v is equal to φ_ε^* on $\partial B_r^+ \setminus \Gamma_r$. Then we get that $v = u$ by uniqueness (see Theorem 4.3). Rescaling back we obtain that the minimizer is φ_ε^*. Now we have – using the (rescaled) energy expansion in (4.50) and (4.51) – that we can express the minimal energy as:

$$E_\varepsilon\left(\varphi_\varepsilon^*; B_r^+\right) = \frac{\pi}{2}\log\frac{r}{\varepsilon} + \frac{\pi}{2}\log 2 + G + \frac{\pi}{2} + O\left(\frac{\log\frac{r}{\varepsilon}}{\frac{r}{\varepsilon}}\right) \tag{7.4}$$

and taking the limit first in ε and then in r we obtain that $\gamma_2 = \gamma_0$. $\square$

We can now prove the main result of this section, analogous to that proved by Ignat and Kurzke [26, Proposition 4.16] [1], whose proof we follow in ours:

Proposition 7.2. *Let u_ε be a sequence of minimizers (or critical points whose energy is that of minimizers up to a constant) of the energy E_ε in Ω. Let $\rho > 0$ be small enough (say less than half of the shortest side-length): then around a corner (which w.l.o.g. we assume to be 0) we have that $u_\varepsilon \to \arg(\cdot)$ in $L^1(B_\rho \cap \partial Q)$ and we have the following second-order lower bound for the energy:*

$$\liminf_{\varepsilon\to 0} \left(E_\varepsilon \left(u_\varepsilon; B_\rho^+\right) - \frac{\pi}{2} \log \frac{\rho}{\varepsilon} \right) \geq \gamma_0. \tag{7.5}$$

Proof. From Theorem 6.3 we have $W^{1,p}$-covergence in the rectangle Ω to a function that is constant on the sides and jumps by $\pm\frac{\pi}{2}$ in the corners - and therefore whose boundary trace near a corner is the (rotated/reflected) argument function. Then $W^{1,p}$-convergence implies by the compactness of the trace operator in Lipschitz domains (see for example [19, Theorem 2.1]) that we have (in particular) L^1 convergence on the boundary to the trace of u_* (as observed above, this is equal to the boundary trace of the argument function in the corner). This proves the convergence claim. For the lower bound we follow closely the proof of Proposition 4.16 in [26]. We start by noticing that the estimate is invariant with respect to rescaling ρ, so it is enough to prove it for $\rho = 1$. In the following we will denote by C_j positive constants which do not depend on $\varepsilon, u_\varepsilon$ or δ. We can assume that there exists a constant $C_0 > 0$ such that

$$\liminf_{\varepsilon\to 0} \left(E_\varepsilon \left(u_\varepsilon; B_1^+\right) - \frac{\pi}{2} \log \frac{1}{\varepsilon} \right) \leq \gamma_0 + C_0, \tag{7.6}$$

since if this is not true then the conclusion of the theorem holds trivially. By considering a subsequence converging to the limit inferior (which always exists)

[1] We only prove it for minimizers – and critical points as in the statement – of the energy, while the proof in [26] is more general. However, since we are only interested in applying this result to minimizers (and minimizers as in the statement), it is enough for our purposes.

we can therefore assume without loss of generality[2] that there exists a constant $C_1 > 0$ such we have for all terms in the sequence (if necessary we discard finitely many terms at the start of the sequence):

$$E_\varepsilon\left(u_\varepsilon; B_1^+\right) \leq \frac{\pi}{2}\log\frac{1}{\varepsilon} + \gamma_0 + C_1. \tag{7.7}$$

The main idea of the proof (as in [26]) is to find a suitable radius where our function is close to the limit function, and to use an interpolation argument to compare their energies.

Step 1: We want to find a radius ρ_* such that $u_\varepsilon\left(\rho_* e^{i\theta}\right)$ has similar properties to the limit function θ.

By Proposition 5.15 we can cover the set S_ε with four balls in the corners of radius $\sigma\varepsilon$ for some $\sigma > 0$, outside of which we have on each side L_k that $|u_\varepsilon - \alpha_k - d_k\pi| \leq \hat{\delta}$ for some $0 < \hat{\delta} < \frac{\pi}{4}, d_k \in \mathbb{Z}$. Then we can use Lemma 5.1 to get the lower bound (see also Remark 2 after Lemma 5.1)

$$\liminf_{\varepsilon\to 0}\left(E_\varepsilon\left(u_\varepsilon; B_{r_0}^+\right) - \frac{\pi}{2}\log\frac{r_0}{\varepsilon}\right) \geq -\tilde{C}, \tag{7.8}$$

for every $r_0 \in \left(\sigma\varepsilon, \frac{1}{2}\right)$. Observe that the constant in Lemma 5.1 only depends on the of square of jump $\frac{1}{2} + d$ (for a minimizer – or a critical point as those considered here – this is always equal to $\pm\frac{1}{2}$, as we remarked in the proof of Theorem 5.16) and a term $(-C\sqrt{\frac{\varepsilon}{\rho}})$ which is bounded from below independently of ρ if $\varepsilon < \rho < R_0$ for a fixed R_0, from which we can conclude that the constant $\tilde{C}$ in (7.8) can be chosen independently of $r_0 \in \left(\varepsilon, \frac{1}{2}\right)$. Combining the two estimates (7.7) and (7.8) we obtain that for a constant $C_2 > 0$ independent of r_0:

$$\limsup_{\varepsilon\to 0} E_\varepsilon\left(u_\varepsilon; B_1^+ \setminus B_{r_0}^+\right) \leq \frac{\pi}{2}\log\frac{1}{r_0} + C_2. \tag{7.9}$$

Let us show this. From (7.7) we can derive the following inequality

[2]Focusing on such a subsequence is not restrictive, as here our goal is to prove a lower bound for the limit inferior.

$$\limsup_{\varepsilon\to 0}\left(E_\varepsilon\left(u_\varepsilon;B_1^+\right)-\frac{\pi}{2}\log\frac{1}{\varepsilon}\right)\leq\gamma_0+C_1. \tag{7.10}$$

Using $E_\varepsilon\left(u_\varepsilon;B_1^+\setminus B_{r_0}^+\right)=E_\varepsilon\left(u_\varepsilon;B_1^+\right)-E_\varepsilon\left(u_\varepsilon;B_{r_0}^+\right)$ we can write:

$$E_\varepsilon\left(u_\varepsilon;B_1^+\setminus B_{r_0}^+\right)-\frac{\pi}{2}\log\frac{1}{r_0}=E_\varepsilon\left(u_\varepsilon;B_1^+\right)-\frac{\pi}{2}\log\frac{1}{\varepsilon}-\left(E_\varepsilon\left(u_\varepsilon;B_{r_0}^+\right)-\frac{\pi}{2}\log\frac{r_0}{\varepsilon}\right)$$

Taking the lim sup on both sides and using (7.8) and (7.10) along with the fact that for a sequence x_n it holds $\limsup_{n\to\infty}(-x_n)=-\liminf_{n\to\infty}x_n$ we then get

$$\limsup_{\varepsilon\to 0}\left(E_\varepsilon\left(u_\varepsilon;B_1^+\setminus B_{r_0}^+\right)-\frac{\pi}{2}\log\frac{1}{r_0}\right)\leq C_1+\gamma+\tilde{C},$$

from which (7.9) follows with $C_2:=C_1+\gamma+\tilde{C}$. Reducing the domain of integration to $B_{1/2}^+\setminus B_{r_0}^+$ (the reason for this will be clear later in the proof) and writing $C_3:=C_2-\frac{\pi}{2}\log\frac{1}{2}>0$ we have the following estimate for the energy:

$$\limsup_{\varepsilon\to 0}E_\varepsilon\left(u_\varepsilon;B_{\frac{1}{2}}^+\setminus B_{r_0}^+\right)\leq\frac{\pi}{2}\log\frac{1}{2r_0}+C_3. \tag{7.11}$$

Now define for $s>0$ the function f_ε as follows, i.e. as the energy restricted to a quarter-circle of radius s:

$$f_\varepsilon(s):=\int_{\partial B_s\cap Q}|\nabla u_\varepsilon|^2d\mathcal{H}^1+\frac{1}{2\pi\varepsilon}\left(\sin^2 u(s,0)+\sin^2\left(u_\varepsilon(0,s)-\frac{\pi}{2}\right)\right), \tag{7.12}$$

so that we have:

$$E_\varepsilon\left(u_\varepsilon;B_r^+\right)=\int_0^r f_\varepsilon(s)\,ds. \tag{7.13}$$

We also define the following sets (where $\delta>0$ will be chosen later):

$$A_\varepsilon:=\left\{s\in\left(0,\frac{1}{2}\right):f_\varepsilon(s)\leq\frac{\frac{\pi}{2}+\delta}{s}\right\}, \tag{7.14}$$

$$G_\varepsilon := \left\{ s \in \left(0, \frac{1}{2}\right) : |u_\varepsilon(s,0)| + |u_\varepsilon(0,s) - \frac{\pi}{2}| < \frac{1}{4} \right\}. \tag{7.15}$$

We now fix r_0 such that

$$r_0 = r_0(\delta) \le \frac{1}{2} e^{-\frac{2C_3}{\delta}},$$

and from now on we consider $\varepsilon < r_0$, without loss of generality.

According to our choice of r_0 we have that:

$$\delta \log \frac{1}{2r_0} \ge 2C_3.$$

The goal is to show that

$$\left[r_0, \frac{1}{2}\right] \cap A_\varepsilon \cap G_\varepsilon \ne \emptyset. \tag{7.16}$$

To this aim we first estimate the measure of $A_\varepsilon \cap \left[0, \frac{1}{2}\right]$. We will show that this is bounded from below by a positive constant; since $|G_\varepsilon| \to \frac{1}{2} - r_0$ as $\varepsilon \to 0$ we will deduce the conclusion from Fatou's lemma. Let $a_\varepsilon := |A_\varepsilon \cap \left[0, \frac{1}{2}\right]|$. Then we can estimate a_ε as follows: since the function $s \mapsto \frac{1}{s}$ is decreasing for $s > 0$ we have

$$\begin{aligned} \frac{\pi}{2} \log \frac{1}{2r_0} + C_3 &\ge \int_{r_0}^{\frac{1}{2}} f_\varepsilon(s)\,ds \ge \int_{\left[r_0, \frac{1}{2}\right] \setminus A_\varepsilon} f_\varepsilon(s)\,ds \\ &\ge \int_{\left[r_0, \frac{1}{2}\right] \setminus A_\varepsilon} \frac{\frac{\pi}{2} + \delta}{s} ds \ge \left(\frac{\pi}{2} + \delta\right) \int_{r_0 + a_\varepsilon}^{\frac{1}{2}} \frac{1}{s} ds \\ &= \left(\frac{\pi}{2} + \delta\right) \log \frac{1}{2\left(r_0 + a_\varepsilon\right)}. \end{aligned} \tag{7.17}$$

From our choice of r_0 and the definition of C_3 we get that

$$-C_3 \ge C_3 - \delta \log \frac{1}{2r_0} \ge \left(\frac{\pi}{2} + \delta\right) \log \frac{r_0}{r_0 + a_\varepsilon}. \tag{7.18}$$

This allows us to finally estimate, for all $\delta \in \left(0, \frac{1}{2}\right)$:

$$a_\varepsilon \geq r_0 \left(e^{\frac{2c_3}{\frac{\pi}{2}+\delta}} - 1 \right) := C_5 r_0 > 0. \tag{7.19}$$

Now choosing a sequence $\varepsilon_n \to 0$ we have $|G_{\varepsilon_n} \cap \left[r_0, \frac{1}{2}\right]| \to \frac{1}{2} - r_0$, so from Fatou's lemma we conclude that

$$\left| \left[r_0, \frac{1}{2} \right] \cap \limsup_{n\to\infty} \left(A_{\varepsilon_n} \cap G_{\varepsilon_n} \right) \right| > 0. \tag{7.20}$$

This shows that there is a radius $\rho_* \in \left[r_0, \frac{1}{2}\right]$ that lies in infinitely many $A_{\varepsilon_n} \cap G_{\varepsilon_n}$. In particular we have that $\rho_* > \varepsilon$.

Step 2: We show that $u_\varepsilon\left(\rho_* e^{i\theta}\right)$ is close to $u^*\left(\rho_* e^{i\theta}\right) := \theta$ in $L^2\left(\partial B_\rho \cap Q\right)$.

Define w_ε to be be the difference between these two functions, i.e.

$$w_\varepsilon(\theta) := u_\varepsilon\left(\rho_* e^{i\theta}\right) - \theta. \tag{7.21}$$

Since $\rho_* \in G_\varepsilon$ we have that $|w_\varepsilon(0)|, |w_\varepsilon\left(\frac{\pi}{2}\right)| < \frac{1}{4}$. Since $s \mapsto \frac{\sin s}{s}$ is decreasing in $\left(0, \frac{1}{4}\right)$ we have for all $\delta \in \left(0, \frac{\pi}{2}\right)$:

$$|w_\varepsilon(\theta)| \leq \frac{4}{\sin\frac{1}{4}} |\sin w_\varepsilon| \leq \frac{4}{\sin\frac{1}{4}} \sqrt{\frac{2\pi^2\varepsilon}{\rho_*}} =: C_7 \sqrt{\frac{\varepsilon}{\rho_*}} \tag{7.22}$$

for $\theta \in \{0, \frac{\pi}{2}\}$. Here we have used that since $\rho_* \in A_\varepsilon$ we have that $|\sin w_\varepsilon|^2 \leq 2\pi\varepsilon f_\varepsilon(\rho_*) \leq 2\pi\varepsilon \frac{\frac{\pi}{2}+\delta}{\rho_*} \leq \frac{2\pi^2\varepsilon}{\rho_*}$.

So we can estimate the L^2 norm of angular derivative of w_ε as follows

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} |\partial_\theta w_\varepsilon(\theta)|^2 d\theta &= \int_0^{\frac{\pi}{2}} \left(|\partial_\theta u_\varepsilon\left(\rho_* e^{i\theta}\right)|^2 - 2\partial_\theta u_\varepsilon\left(\rho_* e^{i\theta}\right) + 1 \right) d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(|\partial_\theta u_\varepsilon\left(\rho_* e^{i\theta}\right)|^2 - 2\partial_\theta w_\varepsilon\left(\rho_* e^{i\theta}\right) - 1 \right) d\theta \\
&= 2\left(w_\varepsilon(0) - w_\varepsilon\left(\frac{\pi}{2}\right) \right) + \int_0^{\frac{\pi}{2}} \left(|\partial_\theta u_\varepsilon\left(\rho_* e^{i\theta}\right)|^2 - 1 \right) d\theta.
\end{aligned}$$

We can estimate the first term from above by $4C_7\sqrt{\frac{\varepsilon}{\rho_*}}$ by (7.22). For the second we use the following estimate:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \left(|\partial_\theta u_\varepsilon\left(\rho_* e^{i\theta}\right)|^2 - 1\right) d\theta &= \int_0^{\frac{\pi}{2}} |\partial_\theta u_\varepsilon\left(\rho_* e^{i\theta}\right)|^2 d\theta - \frac{\pi}{2} \\ &\leq \int_0^{\frac{\pi}{2}} \rho_*^2 \left|\frac{\partial_\theta u_\varepsilon\left(\rho_* e^{i\theta}\right)}{\rho_*}\right|^2 d\theta - \frac{\pi}{2} \\ &\leq \rho_* \int_{\partial B_{\rho_*}\cap Q} |\nabla u_\varepsilon|^2 d\mathcal{H}^1 - \frac{\pi}{2} \\ &\leq \rho_* f\left(\rho_*\right) - \frac{\pi}{2} \leq \delta.\end{aligned}$$

So in conclusion we have

$$\int_0^{\frac{\pi}{2}} |\partial_\theta w_\varepsilon\left(\theta\right)|^2 d\theta \leq \delta + 4C_7\sqrt{\frac{\varepsilon}{\rho_*}}. \tag{7.23}$$

Now we can show that w_ε is small in L^2. For a suitably chosen constant C_8 we have

$$\int_0^{\frac{\pi}{2}} |w_\varepsilon|^2 d\theta \leq \int_0^{\frac{\pi}{2}} \left(w_\varepsilon\left(0\right) + \int_0^\theta \partial_\theta w_\varepsilon\left(\varphi\right) d\varphi\right)^2 d\theta \leq C_8\left(\delta + \sqrt{\frac{\varepsilon}{\rho_*}}\right). \tag{7.24}$$

Step 3: We estimate the energy of an interpolation between u_ε and u^* on a small annulus close to $\partial B_\rho \cap Q$.

Consider the annulus $B_{\rho_*+\eta} \setminus B_{\rho_*}$, for some $\eta > 0$ to be chosen at a later stage: we define an interpolating function $\hat{u}_\varepsilon$ so that $\hat{u}_\varepsilon\left(\rho_*, \theta\right) = u_\varepsilon\left(\rho_* e^{i\theta}\right)$ and $\hat{u}_\varepsilon\left(\rho^* + \eta, \theta\right) = \theta$ as follows:

$$\hat{u}_\varepsilon\left(r, \theta\right) := \theta + \frac{\rho_* + \eta - r}{\eta} w_\varepsilon\left(\theta\right), \text{ for } r \in \left(\rho_*, \rho_* + \eta\right) \text{ and } \theta \in \left(0, \frac{\pi}{2}\right). \tag{7.25}$$

We now want to show that for a suitably chosen η (such that $\eta \to 0$ as $\delta \to 0$) the energy of the interpolating function vanishes in the limit. We can write the energy in polar coordinates as

$$
\begin{aligned}
E_\varepsilon\left(\hat{u}_\varepsilon; B^+_{\rho_*+\eta}\setminus B^+_{\rho_*}\right) &= \int_{\rho_*}^{\rho_*+\eta}\left(\int_0^{\frac{\pi}{2}}\left(\frac{1}{r^2}\partial^2_\theta\hat{u}_\varepsilon+\partial^2_r\hat{u}_\varepsilon\right)rd\theta\right)\\
&+\frac{1}{2\pi\varepsilon}\left(\sin^2\left(\frac{\rho_*+\eta-r}{\eta}w_\varepsilon(0)\right)+\sin^2\left(\frac{\rho_*+\eta-r}{\eta}w_\varepsilon\left(\frac{\pi}{2}\right)\right)\right)dr\\
&=\int_{\rho_*}^{\rho_*+\eta}\left(\int_0^{\frac{\pi}{2}}\left(\frac{1}{r}\left(1+\frac{\rho_*+\eta-r}{\eta}\partial_\theta w_\varepsilon\right)^2+\frac{r}{\eta^2}|w_\varepsilon|^2\right)d\theta\right)\\
&+\frac{1}{2\pi\varepsilon}\left(\sin^2\left(\frac{\rho_*+\eta-r}{\eta}w_\varepsilon(0)\right)+\sin^2\left(\frac{\rho_*+\eta-r}{\eta}w_\varepsilon\left(\frac{\pi}{2}\right)\right)\right)dr.
\end{aligned}
\tag{7.26}
$$

Since $\rho_*\in A_\varepsilon$ we can estimate the part involving the penalty term using the way this set is defined. We have that $|w_\varepsilon(0)|, |w_\varepsilon\left(\frac{\pi}{2}\right)|\leq\frac{1}{4}$ and so we can use the fact that $\sin^2$ is increasing in $(0,\pi/2)$ and that $\sin^2(x)=\sin^2(|x|)$ to estimate the two terms as:

$$
\begin{aligned}
\sin^2\left(\frac{\rho_*+\eta-r}{\eta}w_\varepsilon(0)\right)&\leq\sin^2 w_\varepsilon(0)\\
\sin^2\left(\frac{\rho_*+\eta-r}{\eta}w_\varepsilon\left(\frac{\pi}{2}\right)\right)&\leq\sin^2 w_\varepsilon\left(\frac{\pi}{2}\right),
\end{aligned}
$$

since $0\leq\frac{\rho_*+\eta-r}{\eta}\leq 1$. We can now use the definition of the set A_ε and the fact that $\rho_*\in A_\varepsilon$ to estimate the integral as (recall that $|\sin w_\varepsilon|^2\leq 2\pi\varepsilon f_\varepsilon(\rho_*)$ for $\theta\in\{0,\frac{\pi}{2}\}$):

$$
\begin{aligned}
&\int_{\rho_*}^{\rho_*+\eta}\frac{1}{2\pi\varepsilon}\left(\sin^2\left(\frac{\rho_*+\eta-r}{\eta}w_\varepsilon(0)\right)+\sin^2\left(\frac{\rho_*+\eta-r}{\eta}w_\varepsilon\left(\frac{\pi}{2}\right)\right)\right)dr\\
&\qquad\leq\eta f_\varepsilon(\rho_*)\leq\frac{\eta}{\rho_*}\left(\frac{\pi}{2}+\delta\right).
\end{aligned}
$$

The second term in (7.26) can be estimated using the previous estimate (7.24) on the L^2 norm of w_ε and the fact that we can estimate the integral in r as

$$\int_{\rho_*}^{\rho_*+\eta} \frac{r}{\eta^2} dr = \frac{1}{2} \frac{\eta^2 + 2\rho_* \eta}{\eta^2} \leq 1 + \frac{\rho_*}{\eta}.$$

It now only remains to estimate the first term, which can be done as follows:

$$\begin{aligned}
&\int_{\rho_*}^{\rho_*+\eta} \int_0^{\frac{\pi}{2}} \left(\frac{1}{r} + \frac{2(\rho_* + \eta - r)}{r\eta} \partial_\theta w_\varepsilon + \frac{1}{r} (\partial_\theta w_\varepsilon)^2 \right) d\theta dr \\
&\leq \frac{\pi}{2} \log\left(1 + \frac{\eta}{\rho_*}\right) + \int_{\rho_*}^{\rho^*+\eta} \frac{2}{r} |w_\varepsilon(0) - w_\varepsilon\left(\frac{\pi}{2}\right)| + \frac{1}{r} \left(\delta + 4C_7 \sqrt{\frac{\varepsilon}{\rho_*}}\right) dr \quad (7.27) \\
&\leq \log\left(1 + \frac{\eta}{\rho_*}\right) \left(\frac{\pi}{2} + 8C_7 \sqrt{\frac{\varepsilon}{\rho_*}} + \delta\right),
\end{aligned}$$

where we have used (7.22) and (7.23). Now we can combine these estimates with the inequality $\log(1+t) \leq t$ for $t > 0$ to get an estimate on the energy:

$$\begin{aligned}
E_\varepsilon\left(\hat{u}_\varepsilon; B^+_{\rho_*+\eta} \setminus B^+_{\rho_*}\right) \leq &\log\left(1 + \frac{\eta}{\rho_*}\right) \left(\frac{\pi}{2} + 8C_7 \sqrt{\frac{\varepsilon}{\rho_*}}\right) + \frac{\eta}{\rho_*}\delta \\
&+ C_8 \left(\delta + \sqrt{\frac{\varepsilon}{\rho_*}}\right) \left(\frac{\rho_*}{\eta} + 1\right) + \frac{\pi\eta}{\rho_*}.
\end{aligned} \quad (7.28)$$

Letting $\varepsilon \to 0$ and choosing $\eta = \delta^{\frac{1}{4}} \rho_*$ we get that:

$$\limsup_{\varepsilon \to 0} E_\varepsilon\left(\hat{u}_\varepsilon; B^+_{\rho_*+\eta} \setminus B^+_{\rho_*}\right) \leq \frac{\pi}{2} \log\left(1 + \delta^{1/4}\right) + \delta^{5/4} + C_8 \left(\delta + \delta^{\frac{3}{4}}\right) + \pi \delta^{\frac{1}{4}}. \quad (7.29)$$

Observe that the right hand side tends to 0 as $\delta \to 0$. If we extend $\hat{u}_\varepsilon$ to $B^+_{\rho^*(1+\delta^{1/4})}$ by setting $\hat{u}_\varepsilon := u_\varepsilon$ in $B^+_{\rho_*}$ we can estimate the energy of $\hat{u}_\varepsilon$ from below using the definition of γ_1 in Lemma 7.1, since $\hat{u}_\varepsilon = \varphi^*$ on $\partial B^+_{\rho^*+\eta} \setminus \Gamma_{\rho^*+\eta}$ (recall the definition of $\hat{u}_\varepsilon$ in (7.25)). We have (using $\log \frac{\rho_*(1+\delta^{1/4})}{\varepsilon} = \log \frac{\rho_*}{\varepsilon} + o_\delta(1)$):

$$\liminf_{\varepsilon \to 0} \left(E_\varepsilon\left(\hat{u}_\varepsilon; B^+_{\rho_*(1+\delta^{1/4})}\right) - \frac{\pi}{2} \log \frac{\rho_*}{\varepsilon} \right) \geq \gamma_0 - o_\delta(1). \quad (7.30)$$

Now using the fact that the energy of the interpolating function tends to 0 on the outer annulus $B^+_{\rho_*+\eta} \setminus B^+_{\rho_*}$ as $\delta \to 0$ thanks to (7.29) and that $\hat{u}_\varepsilon = u_\varepsilon$ on $B^+_{\rho_*}$ we obtain that:

$$\liminf_{\varepsilon \to 0} \left(E_\varepsilon \left(u_\varepsilon; B^+_{\rho_*}\right) - \frac{\pi}{2} \log \frac{\rho_*}{\varepsilon} \right) \geq \gamma_0 - o_\delta(1) \tag{7.31}$$

Step 4: we derive an optimal lower bound in the outer annulus $B^+_1 \setminus B^+_{\rho_*}$.

Consider the set

$$S_\varepsilon := \left\{ r \in [\rho_*, 1] : |u_\varepsilon(r,0)| + |u_\varepsilon(0,r) - \frac{\pi}{2}| \leq \frac{1}{4} \right\}. \tag{7.32}$$

Since $u_\varepsilon \to u_*$ in $L^1(B_\rho \cap \partial Q)$, it is clear that $|S_\varepsilon| \to 1 - \rho_*$ as $\varepsilon \to 0$.

Using Hölder's inequality we can derive the following estimate for the L^2 norm of the angular derivative, for $r \in (0,1)$:

$$\begin{aligned} |u_\varepsilon(0,r) - u_\varepsilon(r,0)| &\leq \int_0^{\frac{\pi}{2}} |\partial_\theta u_\varepsilon\left(re^{i\theta}\right)| d\theta \\ &\leq \left(\int_0^{\frac{\pi}{2}} |\partial_\theta u_\varepsilon|^2 \right)^{\frac{1}{2}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}}. \end{aligned}$$

We can now use this to estimate f_ε from below as follows:

$$f_\varepsilon(r) \geq \frac{2}{\pi r} \left(u_\varepsilon(0,r) - u_\varepsilon(r,0)\right)^2 + \frac{1}{2\pi\varepsilon} \left(\sin^2 u_\varepsilon(r,0) + \sin^2 \left(u_\varepsilon(0,r) - \frac{\pi}{2} \right) \right).$$

By the definition of S_ε there exists a constant $C_9 > 0$ such that for all $r \in S_\varepsilon$ we have:

$$\begin{aligned} \sin^2 u_\varepsilon(r,0) + \sin^2 \left(u_\varepsilon(0,r) - \frac{\pi}{2} \right) &\geq 2C_9 \left(u_\varepsilon(r,0)^2 + \left(u_\varepsilon(0,r) - \frac{\pi}{2} \right)^2 \right) \\ &\geq C_9 \left(\frac{\pi}{2} - u_\varepsilon(0,r) + u_\varepsilon(r,0) \right)^2. \end{aligned}$$

From this it follows that:

$$f_\varepsilon(r) \geq \inf_{s\in\mathbb{R}} \left\{ \frac{2s^2}{\pi r} + C_9 \frac{\left(\frac{\pi}{2} - s\right)^2}{2\pi\varepsilon} \right\}. \tag{7.33}$$

Computing the infimum on the right we conclude that f_ε is bounded from below by

$$f_\varepsilon(r) \geq \frac{\pi}{2} \frac{1}{r + C_{10}\varepsilon}, \tag{7.34}$$

for a constant $C_{10} > 0$. Thus we can estimate the energy on this annulus as

$$\begin{aligned} E_\varepsilon\left(u_\varepsilon; B_1^+ \setminus B_{\rho_*}^+\right) &\geq \int_{S_\varepsilon} \frac{\pi}{2} \frac{1}{r + C_{10}\varepsilon} dr \\ &\geq \frac{\pi}{2} \int_{1-|S_\varepsilon|}^{1} \frac{1}{r + C_{10}\varepsilon} dr = \frac{\pi}{2} \log \frac{1 + C_{10}\varepsilon}{1 - |S_\varepsilon| + C_{10}\varepsilon}. \end{aligned} \tag{7.35}$$

Since $|S_\varepsilon| \to 1 - \rho_*$ as $\varepsilon \to$ we obtain that:

$$\liminf_{\varepsilon\to 0} E_\varepsilon\left(u_\varepsilon; B_1^+ \setminus B_{\rho_*}^+\right) - \frac{\pi}{2} \log \frac{1}{\rho_*} \geq 0. \tag{7.36}$$

Combining this with (7.31) we obtain that

$$\liminf_{\varepsilon\to 0} \left(E_\varepsilon\left(u_\varepsilon; B_1^+\right) - \frac{\pi}{2} \log \frac{1}{\varepsilon} \right) \geq \gamma_0 - o_\delta(1) \tag{7.37}$$

and the conclusion follows letting $\delta \to 0$. □

7.2 Second order lower bounds for the full energy

We can now use the results in the previous section to find a second-order lower bound for the full micromagnetic energy (we also include a first order lower bound, coming from our results in Chapter 5):

Theorem 7.3. *Assume that $h \to 0$ and $\eta \to 0$ satisfy the regime*

$$\frac{1}{|\log h|} \ll \varepsilon = \frac{\eta^2}{h|\log h|} \ll 1. \tag{7.38}$$

Assume $\mathbf{m}_h$ *is a sequence of magnetizations such that* $\limsup_{h\to 0} E_h(\mathbf{m}_h) \leq C$ *and define the averaged magnetizations as*

$$\overline{\mathbf{m}}_h(x) := \frac{1}{h}\int_0^h \mathbf{m}(x, x_3)\, dx_3. \tag{7.39}$$

Then we have:

1. ***First order lower bound.*** *The energy satisfies:*

$$\liminf_{h\to 0} E_h(\mathbf{m}_h) \geq 2\pi. \tag{7.40}$$

2. ***Second order energy bound.*** *If we are in the more restrictive regime* $\frac{\log|\log h|}{|\log h|} \ll \varepsilon$ *and the following condition is satisfied*

$$\limsup_{h\to 0} \left(|\log\varepsilon| \left(E_h(\mathbf{m}_h) - 2\pi\right)\right) \leq C, \tag{7.41}$$

then we have the finer energy expansion

$$\liminf_{h\to 0} \left(|\log\varepsilon| \left(E_h(\mathbf{m}_h) - 2\pi\right)\right) \geq W_S(z_k, d_k) + 4\gamma_0, \tag{7.42}$$

where $\gamma_0 =$ *is the constant defined in Lemma 7.1 and* W_S *is the renormalized energy for the* S *state in the rectangle defined in Chapter 6.*

Proof. The proof of the first part follows combining Theorem 2.12 and Lemma 2.11. For the second claim we have that (where $\mathbf{m}_h$ is the magnetization, $\overline{\mathbf{m}}_h$ the average, M_h a unit vector chosen as in Chapter 3 and $\hat{u}_\varepsilon$ a lift of M_h, i.e $M_h = e^{i\hat{u}_\varepsilon}$), thanks to Theorem 2.12 and the results of Chapter 3 (we use the energy estimate of Theorem 3.7) and where z_k denote the vertices of the rectangle Ω:

$$
\begin{aligned}
\liminf_{h\to 0}|\log\varepsilon|\left(E_h\left(\mathbf{m}_h\right)-2\pi\right) &\geq \liminf_{h\to 0}|\log\varepsilon|\left(\overline{E}_h\left(\overline{\mathbf{m}}_h\right)-o\left(\frac{1}{|\log\varepsilon|}\right)-2\pi\right)\\
&= \liminf_{h\to 0}|\log\varepsilon|\left(\overline{E}_h\left(\overline{\mathbf{m}}_h\right)-2\pi\right)\\
&\overset{(\dagger)}{\geq} \liminf_{\varepsilon\to 0}\left(E_{\varepsilon,\eta}\left(m_h\right)-2|\log\varepsilon|\right)\\
&\overset{(*)}{\geq} \left(E\left(M_h\right)-2|\log\varepsilon|\right)\\
&\geq \liminf_{\varepsilon\to 0}\left(E_\varepsilon\left(\hat{u}_\varepsilon\right)-2\pi|\log\varepsilon|\right),
\end{aligned}
$$

where in ($\dagger$) we used (2.72) (which holds for any vector field of length ≤ 1). In ($*$) we use the inequality (3.27) and the fact that the error term in that inequality goes to 0 – this can be seen from the fact that by what we have said after (2.73) we have $E_{\varepsilon,\eta}\left(\overline{m}_h\right)\leq C|\log\varepsilon|$ (since from the assumption that $\limsup_{h\to 0}E_h\left(\mathbf{m}_h\right)\leq C$ we can derive the same kind of bound for the reduced energy, see (2.89) and from this we get the bound for $E_{\varepsilon,\eta}\left(\overline{m}_h\right)$) and by using the estimate $|\log\varepsilon|\leq\log|\log\eta|$ which we can immediately derive from our regime (1.11). We now estimate $E_\varepsilon\left(\hat{u}_\varepsilon\right)$ from below by the energy of a minimizer u_ε and so we get:

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0}\left(E_\varepsilon\left(\hat{u}_\varepsilon\right)-2\pi|\log\varepsilon|\right)\\
&\geq \liminf_{\varepsilon\to 0}\left(E_\varepsilon\left(u_\varepsilon\right)-2\pi|\log\varepsilon|\right)\\
&\overset{(\dagger)}{=} \lim_{\rho\to 0}\liminf_{\varepsilon\to 0}\left(\sum_{k=1}^{4}\left(E_\varepsilon\left(u_\varepsilon;B_\rho\left(z_k\right)\cap\Omega\right)-\frac{\pi}{2}\log\frac{\rho}{\varepsilon}\right)\right.\\
&\left.-2\pi\log\frac{1}{\rho}+E_\varepsilon\left(u_\varepsilon;\Omega\setminus\cup_{k=1}^{4}B_\rho\left(z_k\right)\right)\right)\\
&\geq \lim_{\rho\to 0}\liminf_{\varepsilon\to 0}\sum_{k=1}^{4}\left(E_\varepsilon\left(u_\varepsilon;B_\rho\left(z_k\right)\cap\Omega\right)-\frac{\pi}{2}\log\frac{\rho}{\varepsilon}\right)\\
&+\lim_{\rho\to 0}\liminf_{\varepsilon\to 0}\left(-2\pi\log\frac{1}{\rho}+E_\varepsilon\left(u_\varepsilon;\Omega\setminus\cup_{k=1}^{4}B_\rho\left(z_k\right)\right)\right),
\end{aligned}
\tag{7.43}
$$

Notice that to obtain the equality (†) we have split the domain into the balls of radius ρ in the corners and the rest: since the whole expression does not in fact depend on ρ (since it is equal to $(E_\varepsilon(u_\varepsilon) - 2\pi|\log\varepsilon|)$) we can add the limit for $\rho \to 0$ in front of it without changing anything. Now letting first ε tend to 0 and using the convergence results of Chapter 6 and then letting ρ tend to 0 we can estimate the first term using Proposition 7.2 to estimate the first term and the energy expansion in Chapter 6 to estimate the second term. More precisely we have from Theorem 6.3 that $u_\varepsilon \rightharpoonup u_*$ in $W^{1,p}(\Omega)$ for all $1 \leq p < 2$; observe also that in a corner we have that the boundary trace of u_* conicides with that of the (rotated, reflected) argument function. For fixed $\rho > 0$ we also have (for a subsequence) H^2 weak convergence on $\Omega \setminus \cup_{k=1}^4 B_\rho(z_k)$, i.e. $u_\varepsilon \rightharpoonup u_*$ in $H^2(\Omega \setminus \cup_{k=1}^4 B_\rho(z_k))$. This implies that $u_\varepsilon \to u_*$ in $H^1(\Omega \setminus \cup_{k=1}^4 B_\rho(z_k))$, since $H^2(\Omega \setminus \cup_{k=1}^4 B_\rho(z_k))$ compactly embeds into $H^1(\Omega \setminus \cup_{k=1}^4 B_\rho(z_k))$. We get then that $\int_{\Omega\setminus\cup_{k=1}^4 B_\rho(z_k)} |\nabla u_\varepsilon|^2 dx \to \int_{\Omega\setminus\cup_{k=1}^4 B_\rho(z_k)} |\nabla u_*|^2 dx$. Away from the vortices (i.e. the corners) we have that the penalty term tends to 0 as $\varepsilon \to 0$: indeed we can use the PDE and the H^2 weak convergence (which implies strong L^2 convergence of the normal derivatives of u_ε to the normal derivative of u_* on the boundary away from the corners) to conclude that on any any compact set K on the boundary which is at a positive distance from the corners (since there $\sin^2(u_\varepsilon - g) \leq \frac{1}{4}$):

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_K \sin^2(u_\varepsilon - g) &\leq \frac{C}{\varepsilon}\int_K \sin^2(u_\varepsilon - g)\cos^2(u_\varepsilon - g) = \frac{C}{\varepsilon}\int_K (\sin 2(u_\varepsilon - g))^2 \\
&= C\varepsilon \int_K \left(\frac{\sin 2(u_\varepsilon - g)}{\varepsilon}\right)^2 \leq C\varepsilon \int_K \left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2 \to 0.
\end{aligned}
$$

where $g \equiv \alpha_k$ on the side L_k. Then we have that $E_\varepsilon(u_\varepsilon; \Omega \setminus \cup_{k=1}^4 B_\rho(z_k)) \to \int_{\Omega\setminus\cup_{k=1}^4 B_\rho(z_k)} |\nabla u_*|^2 dx$. We then have from the energy expansion in Proposition 6.5 that:

$$
\lim_{\rho\to 0}\int_{\Omega\setminus\cup_{k=1}^4 B_\rho(z_k)} |\nabla u_*|^2 dx - 2\pi\log\frac{1}{\rho} = W_S(z_k, d_k). \tag{7.44}
$$

The renormalized energy is that for an S state, since that is minimizing. The $W^{1,p}$-convergence of u_ε to u_* implies by the compactness of the trace operator in Lipschitz domains (see for example [19, Theorem 2.1]) that we have L^1 convergence on the boundary to the trace of u_* (as observed above, this is equal to the boundary trace of the argument function in the corner). Since the u_ε are minimizers we can apply Proposition 7.2 to estimate the energy in each ball of radius ρ around the corners. We have for all $k = 1, \dots, 4$ that:

$$\liminf_{\varepsilon \to 0} E_\varepsilon \left(u_\varepsilon, B_\rho \left(z_k\right) \cap \Omega\right) - \frac{\pi}{2} \log \frac{\rho}{\varepsilon} \geq \gamma_0. \tag{7.45}$$

We therefore conclude putting together (7.44) and (7.45) – for each corner – that the last expression in (7.43) must be greater or equal to $4\gamma_0 + W\left(d_k, z_k\right)$, which completes the proof. $\square$

Now constructing a function which near each vertex in ball of a small radius $r > 0$ is chosen as a rescaling in ε of the explicit solution in a corner of Chapter 4 and as u_* (the limit function) in the remaining part of the domain, we can construct functions $\hat{u}_\varepsilon$ which satisfy the equality:

$$\limsup_{\varepsilon \to 0} \left(E_\varepsilon \left(\hat{u}_\varepsilon\right) - 2\pi \log \frac{1}{\varepsilon}\right) = W\left(d_k, z_k\right) + 4\gamma_0. \tag{7.46}$$

This can be seen by a simple calculation which is nearly identical to that in the previous theorem. Hence if we define $\mathbf{M}_h = \left(e^{i\hat{u}_\varepsilon}, 0\right)$ we get the following:

Theorem 7.4. *We can construct a sequence $\mathbf{M}_h \in H^1\left(\mathbf{\Omega}_h; \mathbb{S}^1\right)$ such that*

$$\lim_{h \to 0} E_h\left(\mathbf{M}_h\right) = 2\pi, \tag{7.47}$$

and for which in the more restrictive regime we have the following energy expansion:

$$E_h\left(\mathbf{M}_h\right) = 2\pi + \frac{1}{|\log \varepsilon|}\left(W_S\left(d_k, z_k\right) + 4\gamma_0\right) + o\left(\frac{1}{|\log \varepsilon|}\right), \tag{7.48}$$

where W_S is the renormalized energy of the S state. We can also do the same thing for the C state.

Chapter 8

Multiplicity results for critical points

Kurzke [35] proved that the vortices for minimizers of the functionals (8.1) introduced below are isolated. We will show that under the appropriate energy bound we can extend this result to critical points that are not necessarily minimizers, but still satisfy a logarithmic bound on the energy. For technical reasons we also assume that the penalty term is bounded in ε (i.e. in (8.1) we have $\frac{1}{2\varepsilon}\int_{\partial\Omega}\sin^2(u_\varepsilon - g)\,d\mathcal{H}^1 \leq C$ for some $C > 0$). This assumption is true for minimizers (see for instance [34] on p. 37) and we have reason to believe that it is unnecessary and can be proved for general critical points, but do not pursue this here. It is not needed for vortices on flat parts of the boundary. The strategy we follow, by grouping the vortex points in clusters according to their asymptotic distance, is inspired by a paper by Comte and Mironescu [14]: we will first flatten the boundary, and then through successive blow-ups at different length scales reduce the problem to studying the equilibrium of charges on a line, showing that critical points for the energy lead to critical point of a renormalized energy in the upper half-plane. Using a result by Espin (to be published in a joint paper with the author of this thesis and Kurzke [5]) we will be able to show that in each cluster we only have a single point, thereby proving the single multiplicity results.

8.1 Basics

Before we address the results in this chapter we have to introduce some fundamental concepts that we will employ later. Let Ω be a simply connected domain in $\mathbb{R}^2$ with a C^∞ boundary[1], and define the functionals $E_\varepsilon : H^1(\Omega) \to \mathbb{R}_+$ as

$$E_\varepsilon(u) := \int_\Omega |\nabla u|^2 dx + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(u - g)\, d\mathcal{H}^1, \tag{8.1}$$

where $g : \partial\Omega \to \mathbb{R}$ is a lift of the tangent vector τ, i.e. a function such that $e^{ig} = \tau$, where we identify $\mathbb{R}^2$ with $\mathbb{C}$. These functionals were studied by Kurzke [34, 35] and we refer the reader to these works for more details. Here we will only give the necessary definitions and results. It is easy to show that these functionals do indeed attain their minimum ([35, Proposition 2.1]) and from the functional we can derive the Euler-Lagrange equation satisfied by critical points ([35, Proposition 2.2]):

Proposition 8.1. *Stationary points of* (8.1) *satisfy the following equation:*

$$\int_\Omega \nabla u \cdot \nabla\varphi + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin(2(u-g))\,\varphi = 0 \tag{8.2}$$

for every $\varphi \in H^1(\Omega)$. *Any solution of* (8.2) *is in* $H^2(\Omega)$ *and it is a strong solution of*

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = -\frac{1}{2\varepsilon}\sin(2(u-g)) \quad \text{on} \quad \partial\Omega. \end{cases} \tag{8.3}$$

We now introduce the concept of vortices and recall some basic properties and results shown in [35] (we refer the reader to that paper for more details). We consider a sequence of critical points of E_ε that satisfy an energy bound of the form:

[1]This assumption works, but it might be possible to relax it. To assume that the boundary is $C^{3,\alpha}$ might be enough, but we will not investigate this further.

$$E_\varepsilon\left(u_\varepsilon\right) \leq M \log \frac{1}{\varepsilon}. \tag{8.4}$$

That this bound is reasonable follows from [35, Proposition 3.1], where it is shown that a sequence (u_ε) of minimizers of E_ε satisfies a bound of this kind. We define the *approximate vortex set* S_ε as

$$S_\varepsilon := \left\{x \in \partial\Omega : \sin^2\left(u\left(x\right) - g\left(x\right)\right) \geq \frac{1}{4}\right\}. \tag{8.5}$$

In [35, Section 3] the author shows that for a sequence of critical points satisfying the energy bound (8.4), the approximate vortex set can be covered by ε-balls, whose number is uniformly bounded in ε. In other words there exists N_0 such that for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}, N_\varepsilon \leq N_0$ and points $a_1^\varepsilon, \ldots, a_{N_\varepsilon} \in \partial\Omega$ such that

$$S_\varepsilon \subset \cup_{i=1}^{N_\varepsilon} B_\varepsilon\left(a_i^\varepsilon\right). \tag{8.6}$$

By choosing a subsequence we can assume that N_ε is constant and equal to some $n \in \mathbb{N}$ and by compactness of $\partial\Omega$ we can assume (passing to a further subsequence if necessary) that $a_i^\varepsilon \to a_i^0 \in \partial\Omega$ as $\varepsilon \to 0$. Note that these limit points need not be distinct in principle (i.e. $a_i^0 = a_j^0$ for $i \neq j$ cannot be excluded a priori): that the points are distinct for minimizer was shown by Kurzke [35]: we will show that this remains true for critical points satisfying (8.4).

As we will see we can enlarge these balls (with a radius still of order comparable to ε) so that in these larger balls the value of a critical point u_ε varies rapidly, transitioning from one well of $\sin^2$ to another. These small regions are also called the vortices. In the same paper the author also shows some convergence results for sequences of minimizers, and an energy expansion for minimizers, in terms of a singular term which depends on the number of vortices and a *renormalized energy* which depends on the position of the (limit) vortices. In [35, Theorem 4.2] the author shows that the "jump" near a vortex is of $\pm\pi$, i.e. the vortices are isolated. The analogous result for general critical points satisfying the logarithmic energy bound (8.4) will be the object of this chapter. In Section 8 of the same paper the author shows that critical points of E_ε correspond to critical points of the renormalized energy. We will show a similar result which we will combine with a result by Espin (see [5]) on the equilibrium of ± 1 charges on a line to conclude that vortices must be separated also in this case (i.e. they can only have single multiplicity).

The main idea of our proof is the following: we will proceed by contradiction, assuming that our vortices are not isolated: then by performing a blow-up of the equation on different length scales (corresponding to the mutual distances between the vortices) we obtain several half-space problems, which leads to limit functions with boundary jumps of ± 1 whose position is determined by a renormalized energy in half-space. Then we show that the position of this jumps is a critical point of such a renormalized energy. By a result obtained by Tim Espin on the equilibrium of charges on a line, this turns out to be impossible if we have more than one jump. This will show that on each level we can only have a single point, thus proving that there is a single point overall, which is our claim.

8.2 Some preliminary results

We want to show that the set S_ε can be covered by $\sigma\varepsilon$-balls (for some $\sigma > 0$ independent of ε) that are mutually disjoint, have distance $\gg \varepsilon$ and such that there is a fixed $0 < \delta < \frac{\pi}{4}$ such that $u - g$ outside of each of this balls is close (at distance $< \delta$) to two multiples of π that differ by $\pm\pi$. Consider a sequence of critical points of the energy E_ε that satisfy the energy bound $E_\varepsilon(u_\varepsilon) \leq M|\log\varepsilon|$. We know from the results in [35] that S_ε can be covered by finitely many balls $B_\varepsilon(a_i^\varepsilon)$ for points $a_i^\varepsilon \in \partial\Omega$, where $i = 1, \dots, N_\varepsilon$ where $N_\varepsilon \leq N_0$ for some N_0. Then, choosing a subsequence if necessary, we can assume that $N_\varepsilon = N \leq N_0$ for all ε. Since the boundary $\partial\Omega$ is compact, by further considering a subsequence we can assume that all of these points converge to some limits $a_1, \dots, a_s \in \partial\Omega$. We first show the following gradient bound which will be important later:

Proposition 8.2. *For a family of critical points u_ε of the energy E_ε satisfying the energy bound $E_\varepsilon(u_\varepsilon) \leq M|\log\varepsilon|$ we have the following gradient bound, for a constant $C > 0$ independent of ε:*

$$|\nabla u_\varepsilon| \leq \frac{C}{\varepsilon}. \tag{8.7}$$

Proof. As this will be needed in the proof let us recall that we can assume (by adding a sequence $t_\varepsilon \in 2\pi\mathbb{Z}$ if necessary, see [35, Proposition 5.1]) that $\|u_\varepsilon\|_{L^\infty(\Omega)}$ is bounded uniformly in ε. Consider for each point $x \in \partial\Omega$ the ball $B_\varepsilon(x)$. Then $\bigcup_{x\in\partial\Omega} B_\varepsilon(x)$ is a cover of $\partial\Omega$ and the distance from $\partial\Omega$ of every point in $\Omega \setminus \bigcup_{x\in\partial\Omega} B_\varepsilon(x)$ is $\geq \varepsilon$. For such interior points we can use interior gradient estimates (see for example [18, Theorem 7]) for harmonic functions to conclude that for every $x \in \Omega$ such that $d(x, \partial\Omega) \geq \varepsilon$ the gradient satisfies

$$|\nabla u_\varepsilon(x)| \leq \frac{C}{\varepsilon}, \tag{8.8}$$

for a constant $C > 0$ independent of ε, since $\|u_\varepsilon\|_{L^\infty(\Omega)}$ is uniformly bounded in ε. We now need then to prove a bound of the same kind near the boundary. Let $p \in \partial\Omega$ and let Γ be a compact and connected subset of $\partial\Omega$ which is at a positive distance from p: for $\rho > 0$ small enough we can define $\Gamma := \partial\Omega \setminus B_\rho(p)$. By the Riemann mapping theorem there exists a biholomorphic mapping Φ from the upper half-plane to Ω that sends ∞ to p, and so that $\Phi'(z) \neq 0$ on the boundary of the half-plane. This last thing can be shown by constructing Φ as a composition of the map $T(z) := \frac{z-i}{z+i}$ from the upper half-plane H to the unit disc (notice that the boundary maps to the boundary and the derivative of T is never 0 there) with a conformal transformation S from the unit disc to Ω. We can the use Theorem 3.5 in [49] to show that the derivative of S is never zero on the boundary, and so the same is true for $\Phi = S \circ T : H \to \Omega$. Here we observe that we can extend the derivatives of Φ continuously to the boundary: for T this is obvious, and for S we can use apply the Kellogg-Warschawski theorem (see [49, Theorem 3.6]). Let now Ψ be the inverse of Φ: we observe that Ψ can be also extended continuously to the boundary along with its derivatives away from p and so in particular on Γ. Here we can use for example [6, Theorem A]) to show that S^{-1} extends smoothly to the boundary and then use the fact that T^{-1} also extends smoothly away from $T(\infty) = S^{-1}(p)$. Since Γ is compact and connected, so is $\Psi(\Gamma) \subset \mathbb{R}$, hence it is a compact interval $[a, b]$. Let $I_\Gamma := [a-1, b+1]$. Then on I_Γ the function Φ is continuous and bounded, and so are its derivatives (up to order two in particular). We will prove the bound (8.8) for all points $x \in \bigcup_{y\in\Gamma} B_\varepsilon(y)$. Then, by covering the remaining part with another set and considering a different conformal transformation we obtain the bound for all remaining points. Observe here that there exists $\tilde{R} > 0$ such that $\Psi(\Omega \cap \cup_{y\in\Gamma} B_\varepsilon(y)) \subset B^+_{\tilde{R}}(0)$. In $B^+_{\tilde{R}}$ define a function $w_\varepsilon = (u_\varepsilon - G) \circ \Phi$ (where G is a bounded harmonic extension of g to $\overline{\Phi\left(B^+_{\tilde{R}}\right)}$, which has bounded derivatives up to order 2): this satisfies the following for all smooth test functions φ with compact support in $B^+_{\tilde{R}} \cup \Gamma_{\tilde{R}}$:

$$\int_{B^+_{\tilde{R}}} \nabla w_\varepsilon \cdot \nabla \varphi + \int_{\Gamma_{\tilde{R}}} a \left(\frac{1}{2\varepsilon} \sin 2w_\varepsilon + h \right) \varphi = 0, \tag{8.9}$$

where $a = |\Phi'|$ and $h := \frac{\partial G}{\partial \nu} \circ \Phi$. By what we said we can assume[2] that a, a', h, h' are all bounded on I_Γ; in the following we will write $\|\cdot\|_{L^\infty}$ for the L^∞ norm of any of these functions on I_Γ. Consider a point $z_0 \in \Psi(\Gamma)$[3]: we want to show that for every $x \in B_{R\varepsilon}(z_0)$ we have a gradient bound for w of the form:

$$|\nabla w_\varepsilon(x)| \le \frac{C}{\varepsilon}, \tag{8.10}$$

[2]Notice here that since $\Phi' \neq 0$ we have that a is smooth on the boundary, as a real function from $\mathbb{R} \to \mathbb{R}$.

[3]Observe here that there exists $R > 0$ such that every ball $B_\varepsilon(y), y \in \Gamma$ is contained in $\Phi(B_{R\varepsilon}(z_0))$ for the point $z_0 = \Psi(y)$, and this R can be chosen independently from $y \in \Gamma$, since the maps Φ and Ψ are locally bi-Lipschitz.

for a constant $C > 0$ that does not depend on the point z_0 or ε. We define a rescaled function $v_\varepsilon(x) = w_\varepsilon(\varepsilon x + z_0)$ on $B_{4R}(0)^+$ and so we get the equation for v_ε on $B_{4R}(0)^+$:

$$\int_{B_{4R}^+} \nabla v_\varepsilon \cdot \nabla \varphi + \int_{\Gamma_{4R}} a_\varepsilon \left(\frac{1}{2} \sin 2v_\varepsilon + \varepsilon h_\varepsilon \right) \varphi = 0, \tag{8.11}$$

where $a_\varepsilon(x) = a(z_0 + \varepsilon x)$ and $h_\varepsilon(x) = h(z_0 + \varepsilon x)$. Observe that since a and h are bounded we have that a_ε and h_ε are bounded independently of ε, for all ε small enough. Observe also that a and h are Lipschitz, and therefore so are a_ε and h_ε with a Lipschitz constant independent of ε. The equation (8.11) can also be stated saying that v is a weak solution of:

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{in } B_{4R}^+(0) \\ \frac{\partial v_\varepsilon}{\partial \nu} = -\frac{a_\varepsilon}{2} \sin 2v_\varepsilon - \varepsilon a_\varepsilon h_\varepsilon & \text{on } \Gamma_{4R}. \end{cases} \tag{8.12}$$

We will prove a gradient bound $|\nabla v_\varepsilon| \leq C$ (for a constant independent of ε), which will give the desired bound for w_ε, and hence for u_ε, as we will explain at the end of the proof. To prove the bound for v_ε we use the same approach of Lemma 2.3 in [12]. Define a function q_ε as

$$q_\varepsilon(x, y) = \int_0^y v_\varepsilon(x, t)\, dt. \tag{8.13}$$

Then q_ε satisfies $(\Delta q_\varepsilon)_y = 0$, which implies that Δq_ε is a function of x only, hence it is enough to compute it on $y = 0$. There we have that $\Delta q_\varepsilon = (q_\varepsilon)_{yy} = (v_\varepsilon)_y$. Hence q_ε is a weak solution of

$$\begin{cases} \Delta q_\varepsilon(x, y) = -\frac{a_\varepsilon(x)}{2} \sin 2v_\varepsilon(x, 0) - \varepsilon a_\varepsilon(x) h_\varepsilon(x) & \text{in } B_{4R}^+ \\ q_\varepsilon = 0 & \text{on } \Gamma_{4R}. \end{cases} \tag{8.14}$$

By extending the function via an odd reflection with respect to $y = 0$ we can then get that – since the right-hand side in the first line of (8.14) is in $L^\infty\left(B_{4R}^+\right)$ (indeed we have $\|\Delta q_\varepsilon\|_{L^\infty} \leq K$ for some $K > 0$ independent of ε) –

we have inner $W^{2,p}$ regularity for the extended function and hence $W^{2,p}$ boundary regularity for the problem (8.14) (this is done exactly as in [12, Lemma 2.3], see also [20, Lemma 9.12] for this boundary regularity result). We have that $q_\varepsilon \in W^{2,p}\left(B_{3R}^+\right) \subset C^{1,\beta}\left(\overline{B_{3R}^+}\right)$ for $\beta \in (0,1)$, provided we choose p large enough (for any fixed β). Now, since $\left(q_\varepsilon\right)_y = v_\varepsilon$ we get that $v_\varepsilon \in W^{1,p}\left(B_{3R}^+\right) \subset C^\beta\left(\overline{B_{3R}^+}\right)$ and

$$\|v_\varepsilon\|_{C^\beta\left(\overline{B_{3R}^+}\right)} \leq C, \tag{8.15}$$

where $C > 0$ only depends on R, β and an upper bound for $\|v_\varepsilon\|_{L^\infty}$ (R, β are fixed and $\|v_\varepsilon\|_{L^\infty}$ is bounded uniformly in ε how we noted at the beginning of the proof, hence the constant is independent on ε or the point z_0, since an upper bound for $\|v_\varepsilon\|_{L^\infty\left(B_{4R}^+\right)}$ is given by $\|u_\varepsilon\|_{L^\infty(\Omega)}$). To get a $C^{1,\beta}$ estimate for v_ε, following [12, Lemma 2.3] we need to prove an upper bound for the C^β norm of the right-hand side of the equation (8.14) – we also prove that this bound does not depend on ε. We have:

$$\begin{aligned}&\left|a_\varepsilon(x)\sin 2v_\varepsilon(x,0) + 2\varepsilon a_\varepsilon(x) h_\varepsilon(x) - a_\varepsilon(y)\sin 2v_\varepsilon(y,0) - 2\varepsilon a_\varepsilon(y) h_\varepsilon(y)\right| \\ &\leq \left|a_\varepsilon(x)\sin 2v_\varepsilon(x,0) - a_\varepsilon(y)\sin 2v_\varepsilon(y,0)\right| + 2\varepsilon\left|a_\varepsilon(x) h_\varepsilon(x) - a_\varepsilon(y) h_\varepsilon(y)\right|.\end{aligned} \tag{8.16}$$

The second term can be estimated as

$$2\varepsilon\left|a_\varepsilon(x) h_\varepsilon(x) - a_\varepsilon(y) h_\varepsilon(y)\right| \leq C\varepsilon|x-y|^\beta, \tag{8.17}$$

since the function ah is Lipschitz (and so is $a_\varepsilon h_\varepsilon$) - and therefore Hölder continuous for all exponents $\beta \in (0,1)$ -, where the constant $C > 0$ does not depend on ε. So it only remains to show the estimate for the first term on the right-hand side of (8.16), which we can rewrite as:

$$\left|a_\varepsilon(x)\sin 2v_\varepsilon(x,0) - a_\varepsilon(x)\sin 2v_\varepsilon(y,0)\right| + \left|a_\varepsilon(x)\sin 2v_\varepsilon(y,0) - a_\varepsilon(y)\sin 2v_\varepsilon(y,0)\right|. \tag{8.18}$$

For the first term we have

$$|a_\varepsilon(x)\sin 2v_\varepsilon(x,0) - a_\varepsilon(x)\sin 2v_\varepsilon(y,0)| \leq \|a_\varepsilon\|_{L^\infty}|\sin 2v_\varepsilon(x,0) - \sin 2v_\varepsilon(y,0)|, \tag{8.19}$$

and so we can conclude that (using the fact that v_ε is Hölder continuous and its norm satisfies the bound 8.15)

$$|a_\varepsilon(x)\sin 2v_\varepsilon(x,0) - a_\varepsilon(x)\sin 2v_\varepsilon(y,0)| \leq \|a_\varepsilon\|_{L^\infty}\|v_\varepsilon\|_{C^\beta}|x-y|^\beta \tag{8.20}$$

For the other term we have that for any $\beta \in (0,1)$ there exists some constant $C' > 0$ such that:

$$|a_\varepsilon(x)\sin 2v_\varepsilon(y,0) - a_\varepsilon(y)\sin 2v_\varepsilon(y,0)| \leq |a_\varepsilon(x) - a_\varepsilon(y)| \leq C'|x-y|^\beta, \tag{8.21}$$

since a_ε is Lipschitz continuous (with constant independent of ε) and so Hölder continuous for any exponent $\beta \in (0,1)$. Now, we can conclude from (8.17), (8.20), (8.21) and (8.15)) and the bound $\|\Delta q_\varepsilon\|_{L^\infty} \leq K$ (for K independent of ε as we noticed above) that:

$$\|\Delta q_\varepsilon\|_{C^\beta} \leq \|\Delta q_\varepsilon\|_{L^\infty} + C\varepsilon + \|a\|_{L^\infty}\|v_\varepsilon\|_{C^\beta} + C' \leq \tilde{C}, \tag{8.22}$$

where $\tilde{C} > 0$ is a constant that only depends on R, β (recall that here R and β are fixed) and an upper bound on $\|v_\varepsilon\|_{C^\beta}$ – and hence is independent of ε thanks to (8.15); observe that the upper bound in (8.15) only depends on R, β and an upper bound for $\|v_\varepsilon\|_{L^\infty}$ and therefore we get that this upper bound is also independent of the point $z_0 \in \Psi(\Gamma)$. From this we get using boundary regularity for (8.14) (see for example [20, Theorem 4.11]) that $q \in C^{2,\beta}\left(\overline{B_{2R}^+}\right)$ and hence that $v \in C^{1,\beta}\left(\overline{B_{2R}^+}\right)$. We have that

$$\|v_\varepsilon\|_{C^{1,\beta}\left(B_{2R}^+\right)} \leq C, \tag{8.23}$$

for some constant $C > 0$ depending only on R, an upper bound for $\|v_\varepsilon\|_{L^\infty}$, $\|f'\|_{L^\infty}, \|a\|_{L^\infty}, \|a'\|_{L^\infty}$, hence independent of ε. Since $\|v_\varepsilon\|_{L^\infty} \leq \|u_\varepsilon\|_{L^\infty}$ by construction we see that this bound also does not depend on the point $z_0 \in \Psi(\Gamma)$ around which we did the blow-up. Hence we get the gradient bound for v_ε for a constant C independent of ε and $z_0 \in \Psi(\Gamma)$. By scaling we then get the bound (8.10) for the function w_ε in each ball $B_{R\varepsilon}(x), x \in \Psi(\Gamma)$. To show that the same bound holds for u_ε in $\cup_{y\in\Gamma} B_\varepsilon(y)$ we recall as we observe above that there exists $R > 0$ such that every ball $B_\varepsilon(y), y \in \Gamma$ is contained in $\Phi(B_{R\varepsilon}(x)), x = \Psi(y)$ and we observe that since $w = (u - G) \circ \Phi$ we have

$$\nabla w(x) = (\nabla u - \nabla G)(\Phi(x))\, \Phi'(x), \tag{8.24}$$

and we conclude from the fact that $\Phi' \neq 0$ everywhere (and so on a compact set like $\overline{B_{R\varepsilon}(x)}$ its modulus its bounded away from 0), and from the fact that $|\nabla G| \leq C'$ for a constant $C' > 0$. Thus we have obtained the bound for all points in $\cup_{y\in\Gamma} B_\varepsilon(y)$. Now by covering the remaining part of the boundary $\partial\Omega$ with another set Γ' we can obtain the same bound for all points in $\cup_{y\in\Gamma'} B_\varepsilon(y)$, which concludes the proof. □

We can now prove that the set S_ε can be covered by $\sigma\varepsilon$-balls which are disjoint and whose centres are at a distance asymptotically much larger than ε.

Lemma 8.3. *The set S_ε can be covered by balls $B_{\sigma\varepsilon}(a_i^\varepsilon), i = 1, \ldots, N_0$ for a constant $N_0 \in \mathbb{N}$, such that for $i \neq j$ we have $B_{\sigma\varepsilon}(a_i^\varepsilon) \cap B_{\sigma\varepsilon}(a_i^\varepsilon) = \emptyset$ and $|a_i^\varepsilon - a_j^\varepsilon| \gg \varepsilon$. Furthermore we have that the degree of transition around a point a_i^ε is ± 1 for all i.*

Proof. Thanks to [35, Proposition 3.9] we can cover S_ε with balls $B_\varepsilon(a_i^\varepsilon), i = 1, \ldots, N$ for a constant $N \in \mathbb{N}$. For each i we can consider all indices j such that $\liminf_{\varepsilon\to 0} \frac{|a_i^\varepsilon - a_j^\varepsilon|}{\varepsilon} < +\infty$ and by choosing subsequence we can (in a similar way as we did in the proof of Lemma 5.10) group these balls in larger balls centred at some of points a_i^ε (for a subset of the indices $i = 1, \ldots, N$) of radius $\sigma\varepsilon$ for some $\sigma > 0$). By construction then these balls are disjoint and their centres will be at a distance $\gg \varepsilon$. It remains to prove that the degree of a transition (defined analogously as in Definition 5.11, the only difference here is that instead of α_k we use g, which is not constant) is ± 1 around the points a_i^ε. We can assume (by passing to a subsequence if necessary) that each of these points converge to a limit. Consider all the a_i^ε's that converge to the same limit a_0. Define blow-ups on a scale ε around some a_i^ε as $w_\varepsilon(x) := (u_\varepsilon - G) \circ \Phi(a_i^\varepsilon + \varepsilon x)$, where Φ is a conformal transformation from the half-plane to Ω to flatten the boundary chosen so that $\Phi(0) = a_0$ and $\Phi'(0) = 1$. Then for all $R > 0$ these functions satisfy the equation

$$\int_{B_R^+} \nabla v \cdot \nabla \varphi + \int_{\Gamma_R} a_\varepsilon \left(\frac{1}{2} \sin 2v + h_\varepsilon \right) \varphi = 0, \tag{8.25}$$

where $a_\varepsilon(x) = a(a_i^\varepsilon + \varepsilon x)$ and $h_\varepsilon(x) = \varepsilon h(a_i^\varepsilon + \varepsilon x)$; we have that $a_\varepsilon \to 1$ and $h_\varepsilon \to 0$ locally uniformly. From Proposition 8.7 we get that the gradient ∇w_ε satisfies $|\nabla w_\varepsilon| \le C$ for a constant C. Since by [35, Proposition 5.1] the functions u_ε and hence the functions w_ε can be assumed to be bounded uniformly in ε we can get local uniform convergence for a subsequence by the theorem of Arzelà-Ascoli. We also see that we have boundedness of w_ε in $H^1(B_R^+)$ for all $R > 0$, which implies weak convergence in H^1 on each ball to a function w_*: this is defined in half-space and it is seen easily to be a bounded solution of

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^2_+ \\ \frac{\partial u}{\partial \nu} = -\frac{1}{2} \sin 2u & \text{on} \quad \mathbb{R} = \partial \mathbb{R}^2_+. \end{cases}$$

The solutions of this equation have been classified by Toland (see theorem 8.4 below). In the same way as we did in the proof of Proposition 5.13, since $a_i^\varepsilon \in S_\varepsilon$, by the uniform convergence we can exclude that w_ε converges to a constant, and because we have covered S_ε by finitely many (uniformly in ε) balls, we can also exclude that the limit solution is periodic. Hence there exist $n \in \mathbb{Z}$ and $a \in \mathbb{R}$ and a sign $\sigma \in \{-1, +1\}$ such that

$$u(x, y) = \sigma \arctan \frac{x + a}{y + 1} + \pi n + \frac{\pi}{2}.$$

From this and rescaling we get that the degree of the transition around a_i^ε has to be ± 1. $\square$

Theorem 8.4 (Toland). *Let u be a bounded solution of*

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}_+^2 \\ \frac{\partial u}{\partial \nu} = -\frac{1}{2}\sin 2u & \text{on} \quad \mathbb{R} = \partial \mathbb{R}_+^2. \end{cases}$$

Then u has to satisfy one of the following

1. *u is constant;*

2. *u is periodic;*

3. *There exist $n \in \mathbb{Z}$ and $a \in \mathbb{R}$ and a sign $\sigma \in \{-1, +1\}$ such that*

$$u\,(x, y) = \sigma \arctan \frac{x + a}{y + 1} + \pi n + \frac{\pi}{2}.$$

Remark 5. Notice that the only solutions that have limits at $\pm\infty$ are either constant or monotone.

Consider now a point z_0 which is the limit of vortex points (i.e. the centres a_i^ε of the ε-balls we used to cover the approximate vortex set), and consider a radius $R_0 > 0$ small enough so that the ball $B_{R_0}\,(z_0)$ only contains those points a_i^ε such that $a_i^\varepsilon \to z_0$. We flatten the boundary locally near z_0 via a conformal transformation.

Let $H : \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper-half plane. Let Φ from a set $B_{\rho_0}\,(z_0) \cap \overline{H} \to \overline{\Omega}$ be a bijective conformal transformation such that $\Phi\,(0) = z_0$ and $\Phi'\,(0) = 1$ and such that $\overline{\Omega} \cap B_R\,(z_0) \subset \Phi\left(B_{\rho_0}\,(0) \cap \overline{H}\right)$. Define a function v as

$$v_\varepsilon := (u_\varepsilon - G) \circ \Phi, \tag{8.26}$$

where G is a bounded harmonic extension of g to B_r^+ [4]. Since u_ε is a critical point of E_ε, it follows that v_ε satisfies the following equation in every ball of radius r: [5]:

$$\int_{B_r^+} \nabla v \cdot \nabla \varphi + \int_{\Gamma_r} a(x) \left(\frac{1}{2\varepsilon} \sin 2v + h \right) \varphi = 0, \tag{8.27}$$

for all $\varphi \in H^1(B_r^+)$ which vanish near ∂B_r, where $a(x) = |\Phi'(x)|$ satisfies $a(0) = 1$ and $|a(x) - 1| \leq C|x|$, and where $h := \frac{\partial G}{\partial \nu} \circ \Phi$ is C^∞.

The vortex points a_i^ε will correspond to points $\alpha_i^\varepsilon \in \partial H$ such that $\Phi(\alpha_i^\varepsilon) = a_i^\varepsilon$. We relabel the points $\alpha_1^\varepsilon, \ldots, \alpha_n^\varepsilon \in \mathbb{R}$ if necessary such that $\alpha_i^\varepsilon < \alpha_{i+1}^\varepsilon$ for all $i = 1, \ldots, n-1$. To simplify the notation in what follows we set:

- $\{\alpha_0^\varepsilon, \alpha_{n+1}^\varepsilon\} := \partial B_\rho^+ \cap \{x_2 = 0\}$, where $\alpha_0 < \alpha_{n+1}$ are the preimages of the two points in $\partial\Omega \cap \partial B_R(z_0)$. Then we clearly have that $\alpha_i^\varepsilon < \alpha_{i+1}^\varepsilon$ for all $i = 0, \ldots, n$;

- We call $\Gamma_i^\varepsilon := \left(\alpha_i^\varepsilon + \sigma\varepsilon, \alpha_{i+1}^\varepsilon - \sigma\varepsilon\right)$ for $i = 1, \ldots, n-1$, $\Gamma_0^\varepsilon := (\alpha_0, \alpha_1 - \sigma\varepsilon)$, $\Gamma_n^\varepsilon = (\alpha_n + \sigma\varepsilon, \alpha_{n+1})$ the parts of the flat boundary comprised between two successive points α_i^ε and α_{i+1}^ε;

- We call $\beta_i^\varepsilon := \partial B_{\sigma\varepsilon}(\alpha_i^\varepsilon) \cap \mathbb{R}_+^2$ the small half-circles of radius $\sigma\varepsilon$ around the points α_i^ε in the upper half-plane.

- We call $\beta_\rho := \partial B_\rho(z_0) \cap \mathbb{R}_+^2$ the big outer half-circle around z_0;

- We denote by $d_{ij}^\varepsilon := |\alpha_i^\varepsilon - \alpha_j^\varepsilon|$ the distance between points α_i^ε and α_j^ε, for $i \neq j$.

[4] we note that since g is $C^{1,\alpha}$, by regularity (see [20]) we have G is $C^{1,\alpha}$ near every point on Γ_r.

[5] to make the proof easier to read we drop the subscript ε from v_ε and the function w_ε defined below)

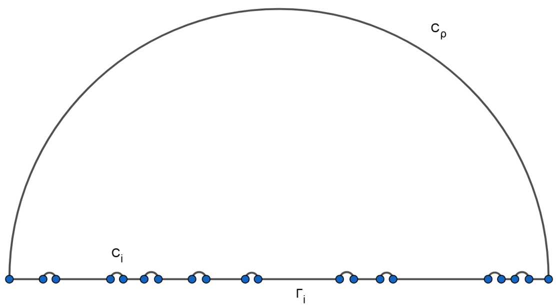


Figure 8.1: The points α_i

- In particular we denote by $l_i^\varepsilon := d_{i,i+1}^\varepsilon$ the distance between two successive points.

To make things easier to read we drop the index ε in the following when this causes no confusion, e.g. we write β_i instead of β_i^ε (see e.g. Figure 8.2)
Observe that if $l_i^\varepsilon \sim l_{i+1}^\varepsilon$, then we also have that $l_{i+1} \sim d_{i,i+2}^\varepsilon \sim l_i$. We now prove a couple of technical results which we will need later on. We start with the following estimate for solutions of the Euler-Lagrange equation away from the bad set, which is similar to [35, Proposition 5.2]:

Lemma 8.5. *Let $u \in H^1(B_R^+)$ satisfying $|u| \leq \arcsin 1/2$ on Γ_R be a solution of:*

$$\int_{B_R^+} \nabla u \cdot \nabla \varphi + \int_{\Gamma_R} a \left(\frac{1}{2\varepsilon} \sin 2u + h \right) \varphi = 0, \tag{8.28}$$

for all $\varphi \in H^1(B_R^+)$ vanishing near ∂B_R, where $a : \Gamma_R \to \mathbb{R}$ is a function satisfying $0 < c \leq a \leq C$. Then for any $\theta < 1$ we have that for a constant K which is independent of u and R and a constant K' which depends on upper bounds for a and h:

$$\int_{B_{\theta R}^+} |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{\Gamma_{\theta R}} u^2 \leq K + K'R. \tag{8.29}$$

Proof. The proof is similar to the proof of [35, Proposition 5.2]. We test the equation with $u\eta^2$, where η is a test function to be chosen later. We have

$$0 = \int_{B_R^+} \eta^2 |\nabla u|^2 + \int_{B_R^+} 2\eta u \nabla u \cdot \nabla \eta + \frac{1}{2\varepsilon} \int_{\Gamma_R} a\,(\sin 2u)\, u\eta^2 + \int_{\Gamma_R} ah\eta^2 u,$$

from which we get by Young's inequality that for every $\alpha > 0$

$$\int_{B_R^+} \eta^2 |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{\Gamma_R} a\,(\sin 2u)\, u\eta^2 \leq \alpha \int_{B_R^+} |\nabla u|^2 \eta^2 + \frac{1}{\alpha} \int_{B_R^+} u^2 |\nabla \eta|^2 + \int_{\Gamma_R} |ah\eta^2 u|,$$

and choosing $\alpha < 1$ we get (using $au \sin 2u \geq m|u|^2$, which follows from the hypotheses on a and the fact that $|u| \leq \arcsin 1/2$) that:

$$\begin{aligned}(1-\alpha) \int_{B_R^+} \eta^2 |\nabla u|^2 + \frac{m}{2\varepsilon} \int_{\Gamma_R} u^2 \eta^2 &\leq (1-\alpha) \int_{B_R^+} \eta^2 |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{\Gamma_R} a\,(\sin 2u)\, u\eta^2 \\ &\leq \frac{1}{\alpha} \int_{B_R^+} u^2 |\nabla \eta|^2 + \int_{\Gamma_R} |ah\eta^2 u|.\end{aligned}$$

Now, choosing η to be a cut-off function satisfying $\eta \equiv 1$ in $B_{\theta R}$ and vanishing outside B_R with $|\nabla \eta| \leq \frac{C}{(1-\theta)R}$, we get the conclusion. □

We can now prove a second order bound which will be useful later:

Proposition 8.6. *Let w_ε be a solution of*

$$\int_{B_R^+} \nabla w_\varepsilon \cdot \nabla \varphi + \int_{\Gamma_R} a_\varepsilon \left(\frac{1}{2\varepsilon} \sin 2w_\varepsilon + h_\varepsilon \right) \varphi = 0, \tag{8.30}$$

such that $\sin^2 w_\varepsilon < \frac{1}{4}$ on Γ_R. Then for every $\theta < 1$ we have

$$\int_{B_{\theta R}^+} |\nabla^2 w_\varepsilon|^2 + \frac{1}{\varepsilon} \int_{\Gamma_{\theta R}} \left| \frac{\partial w_\varepsilon}{\partial \tau} \right|^2 \leq C\,(\theta, R)\,.$$

Proof. We differentiate the equation and test it with the the function $\eta^2 \partial_1 w$, where η is a test function to be chosen later, to get the following equation:

$$\int_{B_R^+} \nabla\partial_1 w \cdot \nabla\left(\eta^2 \partial_1 w\right) + \int_{\Gamma_R} \partial_1 \left(\frac{a \sin 2w}{\varepsilon} + ah\right) \eta^2 \partial_1 w = 0. \tag{8.31}$$

Computing the derivatives, we can write this as:

$$\begin{aligned}\int_{B_R^+} \nabla\partial_1 w \left(2\eta \nabla\eta \partial_1 w + \eta^2 \nabla \partial_1 w\right) + \int_{\Gamma_R} & \frac{\partial_1 a \sin 2w}{\varepsilon} \eta^2 \partial_1 w \\ + \int_{\Gamma_R} & \frac{a 2 \cos 2w \partial_1 w}{\varepsilon} \eta^2 \partial_1 w \\ + \int_{\Gamma_R} & \left(\partial_1 a h + \partial_1 h a\right) \eta^2 \partial_1 w = 0.\end{aligned} \tag{8.32}$$

We can estimate some of the terms using Young's inequality, namely for any $\alpha, \beta, \gamma > 0$:

$$\begin{aligned}\left|\int_{B_R^+} 2\eta\nabla\eta\partial_1 w \nabla\partial_1 w\right| &\le \alpha \int_{B_R^+} \eta^2 |\nabla\partial_1 w|^2 + \frac{1}{\alpha} |\nabla\eta|^2 |\nabla w|^2 \\ \left|\int_{\Gamma_R} \frac{\partial_1 a \sin 2w \eta^2 \partial_1 w}{\varepsilon}\right| &\le \beta \frac{1}{\varepsilon} \int_{\Gamma_R} \eta^2 |\partial_1 w|^2 + \frac{1}{\beta}\frac{1}{\varepsilon} \int_{\Gamma_R} |\partial_1 a \sin 2w|^2 \eta^2 \\ \left|\int_{\Gamma_R} \left(\partial_1 a h + a\partial_1 h\right) \eta^2 \partial_1 w\right| &\le \gamma \frac{1}{\varepsilon} \int_{\Gamma_R} \eta^2 |\partial_1 w|^2 + \frac{1}{\gamma} \int_{\Gamma_R} \left(\partial_1 a h + a \partial_1 h\right)^2 \eta^2.\end{aligned} \tag{8.33}$$

Now, using that $2\cos 2w = 2(1 - 2\sin^2 w) \ge 1$, and the fact that a is bounded from below by a positive constant and using the estimates above we can write that (choosing α, β and γ small enough), by choosing η to be a cut-off function that is 0 outside of B_R^+, equal to 1 in $B_{\theta R}^+$ and whose gradient satisfies $|\nabla\eta| \le \frac{C}{R(1-\theta)}$:

$$\begin{aligned}\int_{B_R^+} \eta^2 |\nabla\partial_1 w|^2 + \frac{c}{\varepsilon} \int_{\Gamma_R} \eta^2 |\partial_1 w|^2 \le & \frac{1}{\alpha} \int_{B_R^+} |\nabla\eta|^2 |\nabla w|^2 + \frac{1}{\beta}\frac{1}{\varepsilon} \int_{\Gamma_R} |\partial_1 a \sin 2w|^2 \eta^2 \\ & + \frac{1}{\gamma} \int_{\Gamma_R} \left(\partial_1 a h + a \partial_1 h\right)^2 \eta^2.\end{aligned} \tag{8.34}$$

We now get the conclusion by Lemma 8.5.

□

8.3 Critical points have single jumps

We are now ready to begin the proof of the main result of this chapter, namely that vortices are isolated, expressed by Theorem 8.15 below. This says that vortices are isolated for critical points satisfying the energy bound (8.4) and it extends the analogous result for minimizers by Kurzke [35]. The proof we give combines the results and techniques already known for the isolated vortices case with blow-up techniques inspired by those of Comte and Mironescu [14] and with a crucial result for charges on a line, namely Theorem 8.21. Let u_ε be a sequence of critical points of E_ε satisfying the logarithmic energy bound $E_\varepsilon(u_\varepsilon) \leq M|\log \varepsilon|$. We know that the approximate vortex set S_ε can be covered by finitely many disjoint $\sigma\varepsilon$-balls (see [35, Corollary 6.3]), which means that we have, for points $a_i^\varepsilon \in \partial\Omega$, $i = 1, \ldots, N_\varepsilon$, that

$$S_\varepsilon \subset \bigcup_{i=1}^{N_\varepsilon} B_{\sigma\varepsilon}\left(a_i^\varepsilon\right),$$

where N_ε is uniformly bounded in ε (see [35, Corollary 6.3]). The centres a_i^ε of these balls converge to points a_i^0. We call $\{a_1, \ldots, a_s\} \subset \partial\Omega$ the set of the distinct limit points (note that, if $i \neq j$, the sequences a_i^ε and a_j^ε can in principle converge to the same limit, i.e. $a_i^0 = a_j^0$: the results in this chapter will show that this is indeed not the case for critical points satisfying the prescribed logarithmic energy bound). Let z_0 be one of such limit points and let w.l.o.g. $\{a_1^\varepsilon, \ldots, a_n^\varepsilon\}$ be the set of points in $\{a_i^\varepsilon, i = 1, \ldots, N_\varepsilon\}$ that converge to z_0, and α_i^ε be defined as above as their preimages under the conformal map Φ. Set $\mathcal{A} := \{1, \ldots, n\}$.

We have already noticed above that the points α_i have an asymptotic distance to each other that is large compared [6] to ε, i.e. $|\alpha_i^\varepsilon - \alpha_j^\varepsilon| \gg \varepsilon$ for every $i \neq j$. We recall that we numbered the points $\{\alpha_1^\varepsilon, \dots, \alpha_n^\varepsilon\}$ such that $\alpha_i < \alpha_{i+1}$ – where we identify $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$ with $\mathbb{R}$ (see Figure 8.2). Define the set $\mathcal{P} := \{(i,j) \in \mathcal{A} \times \mathcal{A} : i < j\}$. Then we can define an equivalence relation on $\mathcal{P}$ as follows: $(i,j) \sim (i', j')$ if and only if $|\alpha_i^\varepsilon - \alpha_j^\varepsilon| \sim |\alpha_{i'}^\varepsilon - \alpha_{j'}^\varepsilon|$. The following is the equivalent of [14, Lemma 2]:

Lemma 8.7. *Let $\varepsilon_n \to 0$. Then there exist a subsequence (which we still denote by ε_n) and constants C, N such that there exist sequences $\lambda_k^{\varepsilon_n} \to 0, k = 1, \dots, N$, a partition $\{\mathcal{L}_1, \dots, \mathcal{L}_N\}$ of $\mathcal{P}$ such that:*

- $\varepsilon_n \ll \lambda_k^{\varepsilon_n} \ll 1$ *for all* $k = 1, \dots, N$;
- $\lambda_k^{\varepsilon_n} \ll \lambda_{k+1}^{\varepsilon_n}$ *for all* $k = 1, \dots, N-1$;
- $(i,j) \in \mathcal{L}_k$ *if and only if* $\lambda_k^{\varepsilon_n} \leq |a_i^{\varepsilon_n} - a_j^{\varepsilon_n}| \leq C\lambda_k^{\varepsilon_n}$.

Proof. The proof is completely analogous to that of [14, Lemma 2], and relies on the fact that the number of bad discs is uniformly bounded in ε. □

For every $i \in \mathcal{A}$ and for any asymptotic distance in the scale determined by Lemma 8.7 we define as in the proof of [14, Lemma 4] a set of the indices in $\mathcal{A}$ corresponding to points that have asymptotic distance from $a_i^{\varepsilon_n}$ which is at most $\lambda_k^{\varepsilon_n}$.

$$\mathcal{L}_{i,k} := \{j \in \mathcal{A} : |a_i^{\varepsilon_n} - a_j^{\varepsilon_n}| \leq C\lambda_k^{\varepsilon_n}\}.$$

Clearly we have

$$\mathcal{L}_{i,k} = \{i\} \cup \{j \in \mathcal{A} : (i,j) \in \mathcal{L}_s \quad \text{for} \quad s \leq k\}.$$

Our goal is then to show that all of these sets just consist of a single element.

[6] Notice that the distance between the points remains much larger than ε even through the conformal transformation, since this is bi-Lipschitz.

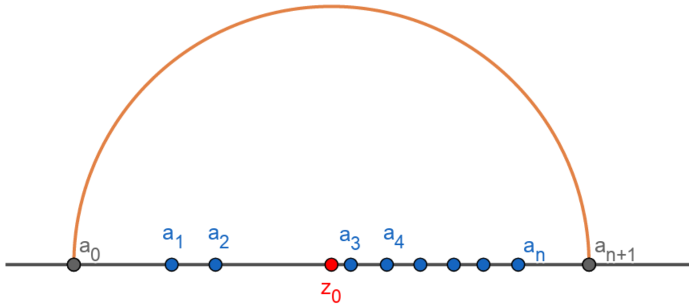


Figure 8.2: The points a_i

Observe that for all $R > 0$ large enough the set $\{\alpha_j^{\varepsilon_n} : j \in \mathcal{L}_{i,1}\}$ is contained the ball $B_{R\lambda_1^\varepsilon}(\alpha_i^{\varepsilon_n})$ for n sufficiently large (say $n \geq N_0$ for some $N_0 \in \mathbb{N}$), and that no other point $\alpha_k^{\varepsilon_n}$ is contained in this ball, i.e. $B_{R\lambda_1^\varepsilon}(a_i^{\varepsilon_n}) \bigcap \{a_k^{\varepsilon_n} : k \in \mathcal{A} \setminus \mathcal{L}_{i,1}\} = \emptyset$. Also observe that for all $k, j \in \mathcal{L}_{i,1}$ we have $|\alpha_i^\varepsilon - \alpha_j^\varepsilon| \geq c\lambda_1^\varepsilon$, for some constant $c > 0$ thanks to the last point in Lemma 8.7, since λ_1 is the smallest scale.

Let $\varepsilon' = \varepsilon_1' := \varepsilon/\lambda_1^\varepsilon$ and define functions $w_{\varepsilon'}$ in B_R^+ as

$$w_{\varepsilon'}(z) := v_\varepsilon(\lambda_1^\varepsilon z), \tag{8.35}$$

where $v_\varepsilon(z) = (u_\varepsilon - G) \circ \Phi(z + \alpha_i)$. We get a sequence of harmonic function, which satisfy the equation

$$\int_{B_r^+} \nabla w_{\varepsilon'} \cdot \nabla \varphi + \int_{\Gamma_r} a_{\varepsilon'} \left(\frac{1}{2\varepsilon'} \sin 2w_{\varepsilon'} + h_{\varepsilon'} \right) \varphi = 0,$$

where $a_{\varepsilon'}(x) = a\left(\frac{\varepsilon}{\varepsilon'}x + \Phi^{-1}(a_i^\varepsilon)\right) = a\left(\frac{\varepsilon}{\varepsilon'}x + \alpha_i^\varepsilon\right)$ and similarly we define $h_{\varepsilon'}(x) = \frac{\varepsilon}{\varepsilon'}h\left(\frac{\varepsilon}{\varepsilon'}x + \Phi^{-1}(a_i^\varepsilon)\right)$. We have that $a_{\varepsilon'} \to 1$ and $h_{\varepsilon'} \to 0$ locally uniformly (recall that $\frac{\varepsilon}{\varepsilon'} = \lambda_1^\varepsilon \to 0$) – so in particular we have uniform convergence on Γ_r for any $r > 0$.

By [35, Proposition 5.1] we have that the sequence u_ε can be assumed to be bounded in L^∞, so the same will hold for v_ε and hence for w_ε. Now using Lemma 8.5 we can show that w_ε is bounded in $W^{1,p}\left(B_R^+\right)$, for every $1 \leq p < 2$, for any $R > 0$. This is the statement of Theorem 8.9 below. The proof is the same as in [35, Theorem 5.4] and follows from the L^2 estimate for the gradient given by Lemma 8.5, and needs the following preliminary result:

Proposition 8.8. *Let the functions w_ε be as above. Let $\sigma > 0$ be such that the balls $B_\sigma\left(\beta_k^\varepsilon\right)$, where $\beta_k^\varepsilon := \left(\alpha_k^\varepsilon - \alpha_i^\varepsilon\right)/\lambda_1^\varepsilon$, are mutually disjoint[7]. Define $B_\sigma^R := B_R^+ \setminus \cup_k B_\sigma\left(\beta_k^\varepsilon\right)$ and $\Gamma_\sigma^R = \partial\mathbb{R}_+^2 \cap B_R \setminus \cup_k B_\sigma\left(\beta_k^\varepsilon\right)$. Then if we define F_ε on any set $A \subset \overline{\mathbb{R}_+^2}$ as*

$$F_\varepsilon\left(u, A\right) := \int_A |\nabla u|^2 dx + \frac{1}{2\varepsilon}\int_{\overline{A}\cap\partial\mathbb{R}_+^2} \sin^2 u d\mathcal{H}^1, \tag{8.36}$$

we can estimate the energy of w_ε on B_R^σ as:

$$F_\varepsilon\left(w_\varepsilon, B_\sigma^R\right) \leq C \log\frac{1}{\sigma}, \tag{8.37}$$

for a constant $C > 0$ that does not depend on ε.

Proof. The proof is the same as in [35, Proposition 5.3] and relies on covering the part near the boundary with a number of balls which is logarithmic in σ and using the estimate of Lemma 8.5. In the rest of the domain we use classical interior gradient bounds to get a bound which is also logarithmic in σ, thus concluding the proof. We refer to the aforementioned proof for more details. □

We can now prove the desired $W^{1,p}$ bounds:

Theorem 8.9. *Let $w_{\varepsilon'}$ be defined as above. Then for all $1 \leq p < 2$ there exists a constant $C = C\left(R, p\right) > 0$ such that:*

[7] Such a σ exists since by construction we have $|\beta_i^\varepsilon - \beta_j^\varepsilon| \geq c$ for a constant c.

$$\int_{B_R^+} |\nabla w_{\varepsilon'}|^p \leq C. \tag{8.38}$$

Proof. The proof is the same as in [35, Theorem 4.2], so we will not repeat it here, but we remark that it crucially uses the estimate (8.37) and a suitable decomposition of B_R^+. We refer to [35, Theorem 4.2] for more details. $\square$

Now we can prove an important convergence result:

Proposition 8.10. *For every $R > 0$ and every $1 \leq p < 2$ we have that $w_{\varepsilon'} \rightharpoonup w_*$ in $W^{1,p}\left(B_R^+\right)$, where $w_* = \sum_i d_k \arg\left(z - \beta_k\right)$ on the boundary, for some $d_k \in \mathbb{Z}$.*

Proof. We start by remarking that in order for this to work it is necessary that p is strictly less than 2. Boundedness of $w_{\varepsilon'}$ in $W^{1,p}\left(B_R^+\right)$ gives now weak-* convergence[8] $w_{\varepsilon'} \rightharpoonup w_*$ in $W^{1,p}\left(B_R^+\right)$, by taking a subsequence if necessary, to some w_*. Since we can do this for every $R > 0$, we get that the limit w is a harmonic function defined in the upper half-plane, i.e. we have that for every $R > 0$ the sequence $w_{\varepsilon'}$ converges weakly in $W^{1,p}\left(B_R^+\right)$ for all $p < 2$ to a function w_* defined in $\mathbb{R}^2_+$. We now want to show that this function satisfies:

$$\begin{cases} \Delta w_* = 0 & \text{in } \mathbb{R}^2_+ \\ w_* = \sum_i d_k \arg\left(z - \beta_k^{(i)}\right) & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

where $\beta_k^{(i)} := \lim_{\varepsilon\to 0}\left(\alpha_k^\varepsilon - \alpha_i^\varepsilon\right)/\lambda_1^\varepsilon$ (passing to a subsequence if necessary) and $d_k \in \mathbb{Z}$. Let $\sigma > 0$ be small enough, say $\sigma < \frac{1}{2}\min_{k\neq j}|\beta_k^{(i)} - \beta_j^{(i)}|$ and let $\Gamma_\sigma^R := \partial\mathbb{R}^2_+ \cap B_R \setminus \cup_k B_\sigma\left(\beta_k^{(i)}\right)$, for $R > 0$ large enough, say $R > 2\max_k|\beta_k^{(i)}|$. We start by showing that

$$\limsup_{\varepsilon'\to 0} \frac{1}{\varepsilon'} \int_{\Gamma_\sigma^R} \sin^2 w_{\varepsilon'} = 0. \tag{8.39}$$

We have (since $\sin^2 w_{\varepsilon'} \leq \frac{1}{4}$ on Γ_σ^R):

---

[8]In this case by reflexivity the notions of weak and weak-* convergence are equivalent.

$$
\begin{aligned}
\frac{1}{\varepsilon'}\int_{\Gamma_\sigma^R}\sin^2 w_{\varepsilon'} &\le \frac{C}{\varepsilon'}\int_{\Gamma_\sigma^R}\sin^2 w_{\varepsilon'}\cos^2 w_{\varepsilon'} = \frac{C}{\varepsilon'}\int_{\Gamma_\sigma^R}(\sin 2w_{\varepsilon'})^2 \\
&= C\varepsilon'\int_{\Gamma_\sigma^R}\left(\frac{\sin 2w_{\varepsilon'}}{\varepsilon'}\right)^2 = C\varepsilon'\int_{\Gamma_\sigma^R}\frac{4}{a_{\varepsilon'}^2}\left(\frac{a_{\varepsilon'}\sin 2w_{\varepsilon'}}{2\varepsilon'}\right)^2 \qquad (8.40)\\
&\le C\varepsilon'\int_{\Gamma_\sigma^R}\left(\frac{\partial w_{\varepsilon'}}{\partial\nu} - h_{\varepsilon'}a_{\varepsilon'}\right)^2 \to 0,
\end{aligned}
$$

where we have used that $a_{\varepsilon'} \to 1, h_{\varepsilon'} \to 0$ locally uniformly (and so in particular $a_{\varepsilon'} > \frac{1}{2}$ for ε' small enough) and the fact that the H^2 bounds in Proposition 8.6 imply weak H^2 convergence away from the vortices (hence on Γ_σ^R), which in turn implies L^2 (and so also L^1) convergence for the normal derivatives on the boundary. This implies that the limit function will satisfy $\sin^2 w_* = 0$ on Γ_σ^R. Since outside of balls of radius ε' around β_k^ε the functions $w_{\varepsilon'}$ are close to a unique number in $\pi\mathbb{Z}$ on each connected component and $\beta_k^{(i)} := \lim_{\varepsilon\to 0}\beta_k^\varepsilon$ we can see that the limit does not in fact depend on σ and R. Thus the limit function w_* is then given by $w_* = \sum_k d_k \arg\left(z - \beta_k^{(i)}\right)$ on the boundary, for some $d_k \in \mathbb{Z}$. We indeed have that this is the expression for the function overall, by uniqueness of bounded solutions to the Dirichlet problem on a half-plane with step functions as boundary values. We show this in Proposition 8.11. ☐

Proposition 8.11. *Let v be a bounded harmonic function on the upper half-plane H whose boundary value is a step function, i.e. a piecewise constant function on $\mathbb{R}$ which jumps at finitely many points $b_1, \ldots, b_n \in \mathbb{R}$ by numbers $d_1, \ldots, d_n \in \mathbb{R}$ and is continuous on $\overline{H} \setminus \{b_1, \ldots, b_n\}$. Then v is uniquely determined by the points b_i and the jumps d_i.*

Proof. In the following we identify the real plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Let v and u be two harmonic functions that satisfy the given boundary condition, and let $w = u - v$. Then w is bounded, harmonic in H and continuous on $\overline{H} \setminus \{b_1, \ldots, b_n\}$, and equal to 0 on the boundary $\{\Im(z) = 0\}$. Let w' be a harmonic conjugate of w on H. Then w' is continuous on $\overline{H} \setminus \{b_1, \ldots, b_n\}$.

Define $F = w + iw'$. Extend F to $\{\Im(z) < 0\}$ as $F(\overline{z}) = \overline{F(\overline{z})}$. By the Schwarz reflection principle we obtain a bounded function which is holomorphic in $\mathbb{C} \setminus \{b_1, \ldots, b_n\}$ and whose real part is bounded near the points b_i: this implies that they are removable singularities. Indeed, the only other two possibilities are that these are poles or essential singularities. In the former case, we can proceed as follows: if b_i is a pole of F, it means it is a zero of $G = 1/F$ and that G has a removable singularity at b_i and is holomorphic in a neighbourhood of it. Since G is not identically zero and it is an open function (by the Open mapping theorem), the image of a disk $D_\delta(b_i)$ centred at b_i will contain a disk $D_r(0)$, i.e. . Hence for the punctured disk without b_i we get

$$F\left(D_\delta(b_i) \setminus \{b_i\}\right) = \frac{1}{G\left(D_\delta(b_i) \setminus \{b_i\}\right)} \supset \frac{1}{D_r(0) \setminus \{0\}} = \mathbb{C} \setminus \overline{D_{1/r}(0)}.$$

This shows that, near a pole, both the real and imaginary parts must be unbounded. If b_i is an essential singularity we get the same conclusion by the Casorati-Weierstrass theorem. Hence we conclude that the b_i's must be removable singularities, since near those points the real part u is bounded. Hence F can be extended to a holomorphic function on all of $\mathbb{C}$ with bounded real part. Then e^F is a bounded holomorphic function, which must be constant by Liouville's theorem. Then F is also constant, and so must therefore be w. Since $w = 0$ on $\{\Im(z) = 0\}$, we conclude that $w = 0$ everywhere. This concludes the proof of uniqueness. $\square$

Remark 6. We remark that the assumption that the solution is bounded is necessary to obtain uniqueness. Otherwise if u is a solution, then so is $w = u + cy$ for any $c \in \mathbb{R}$.

Our next step is to show that the point $\beta := (\beta_1, \ldots, \beta_t)$ is a critical point for a suitable half-space renormalized energy. We first show an energy expansion for w, from which we will get the right renormalized energy:

Proposition 8.12. *Let* $v = \sum_k d_k \arg(z - \beta_k)$ *be defined on* $\overline{\mathbb{R}^2_+} \setminus \{\beta_i\}$*, for* $\beta_i \in \partial\mathbb{R}^2_+$ *and let* $\rho < \frac{1}{2}\min_{i\neq j}|\beta_i - \beta_j|$*. Let* $R > \max\{|\beta_i|, \rho\}$*. Then there exist* $\rho_0, R_0, C_1, C_2 > 0$ *such that for every* $R > R_0, \rho < \rho_0$ *we have that*

$$\left|\int_{B_R^+ \setminus \bigcup_k B_\rho(\beta_k)} |\nabla v|^2 - \left(-\pi\sum_k d_k^2 \log\rho + \pi\left(\sum_k d_k\right)^2 \log R + W(\beta_k, d_k)\right)\right|$$
$$\leq C_1 \rho\log\rho + C_2 \log\frac{\log R}{R},$$

where $W(\beta_k, d_k) = -\pi\sum_{i\neq j} d_j d_i \log|\beta_i - \beta_j|$*.*

Proof. The proof is the same as in chapter 6, see Proposition 6.4. □

The following theorem shows that critical points of the energy functionals lead in the limit to a critical point of the half-space renormalized energy after blow-up:

Theorem 8.13. *Let* u_ε *be a sequence of critical points of* E_ε *satisfying the logarithmic energy bound* $E_\varepsilon(u_\varepsilon) \leq M\log\frac{1}{\varepsilon}$ *and so that the penalty term satisfies* $\frac{1}{2\varepsilon}\int_{\partial\Omega}\sin^2(u_\varepsilon - g)\, d\mathcal{H}^1 \leq C$ *for some* $C > 0$*. Then the point* $b = (b_k)$ *obtained as above is a critical point for* W*.*

Proof. We follow an approach which is similar to that which Jerrard [30] used to prove an analogous result for the Ginzburg-Landau functionals (this result was already shown by Bethuel, Brezis, Hélein in their book [7]). We first show the result assuming the boundary is already flat: this makes it easier to follow. Below we then present the proof for a non-flat boundary which we flatten via a conformal transformation. Let u_* be the limit function as above, i.e. $u_*(z) = \sum_k d_k \arg(z - b_k)$. Now define the *stress-energy tensor* S_* as

$$S_* = \nabla u_* \otimes \nabla u_* - \frac{1}{2}|\nabla u_*|^2 Id. \tag{8.41}$$

Fix some index i. Let φ be a smooth compactly supported function in $\overline{\mathbb{R}^2_+}$ such that $\operatorname{supp}(\varphi) \cap \{b_k\}_k = \{b_i\}$, such that $\nabla\varphi \equiv c$ in a neighbourhood of b_i for a constant vector c, and such that $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial \mathbb{R}^2_+$. Observe that this implies that near b_i we have $ce_1 = \nabla \varphi$, since this is true on the boundary, given that the normal derivative is zero. We will prove the following identities, which will show the desired conclusion due to the arbitrariness of φ:

$$0 = \int_{\mathbb{R}^2_+} \nabla \nabla \varphi : S_* = \nabla \varphi \cdot \nabla_{b_i} W(b). \tag{8.42}$$

To show the first equality we proceed as follows: for $\varepsilon > 0$ and for a sequence of critical points u_ε converging to u_* as before (and which satisfies the usual H^2 bounds away from the vortices) define the matrix valued measures:

$$\hat{S}_\varepsilon := \left(\nabla u_\varepsilon \otimes \nabla u_\varepsilon - \frac{1}{2} |\nabla u_\varepsilon|^2 Id \right) d\mathcal{L}^2 - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\varepsilon} \Psi(u_\varepsilon) \, d\mathcal{H}^1 \llcorner \mathbb{R}, \tag{8.43}$$

where $\Psi(t) = \frac{1}{2} \sin^2 t$. Define S_ε as the corresponding stress-energy tensor

$$S_\varepsilon := \nabla u_\varepsilon \otimes \nabla u_\varepsilon - \frac{1}{2} |\nabla u_\varepsilon|^2 Id. \tag{8.44}$$

Then we have that $\operatorname{div} S_\varepsilon = \Delta u_\varepsilon \nabla u_\varepsilon = 0$, since u_ε is harmonic. Now we get:

$$\begin{aligned}
\int_{\overline{\mathbb{R}^2_+}} \nabla \nabla \varphi : \hat{S}_\varepsilon &= \int_{\mathbb{R}^2_+} \nabla \nabla \varphi : S_\varepsilon - \frac{1}{\varepsilon} \int_{\mathbb{R}} \varphi_{xx} \Psi(u_\varepsilon) \\
&= -\int_{\mathbb{R}^2_+} \nabla \varphi \cdot \operatorname{div} S_\varepsilon + \int_{\mathbb{R}} \nu \cdot S_\varepsilon \nabla \varphi + \frac{1}{\varepsilon} \int_{\mathbb{R}} \varphi_x \Psi'(u_\varepsilon) \partial_x u_\varepsilon \\
&= \int_{\mathbb{R}} \frac{\partial u_\varepsilon}{\partial \nu} \nabla u_\varepsilon \cdot \nabla \varphi - \frac{1}{2} \frac{\partial \varphi}{\partial \nu} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} \int_R \varphi_x \Psi'(u_\varepsilon) \partial_x u_\varepsilon \\
&= \int_{\mathbb{R}} \left(\frac{\partial u_\varepsilon}{\partial \nu} + \frac{1}{\varepsilon} \Psi'(u_\varepsilon) \right) \partial_x u_\varepsilon \varphi_x = 0,
\end{aligned} \tag{8.45}$$

because of the boundary condition satisfied by u_ε. This shows that $\int_{\overline{\mathbb{R}^2_+}} \nabla \nabla \varphi : \hat{S}_\varepsilon = 0$. Now we want to show that these integrals converge to $\int_{\mathbb{R}^2_+} \nabla \nabla \varphi : S_*$, which will imply that this last integral is also equal to zero. We notice that $\operatorname{supp} \nabla \nabla \varphi$ is away from the vortices. Away from the vortices thanks to the bounds in Proposition 8.6 we have weak H^2 convergence of u_ε to u_* – which implies strong L^2-convergence of the gradients, by the Kondrachov embedding theorem (see [1, Theorem 6.3, Part I]) – and the convergence of the penalty term to zero, as was shown above in (8.40). This shows the convergence of $\int_{\overline{\mathbb{R}^2_+}} \nabla\nabla\varphi : \hat{S}_\varepsilon$ to $\int_{\mathbb{R}^2_+} \nabla\nabla\varphi : S_*$ as $\varepsilon' \to 0$ (here we use the fact that the integral is in fact only on $\operatorname{supp}(\nabla\nabla\varphi)$), and therefore the claim. So what is left to show to get the conclusion is that

$$\int_{\mathbb{R}^2_+} \nabla\nabla\varphi : S_* = \pi \nabla W_{b_i}(b) \cdot \nabla\varphi, \tag{8.46}$$

since the arbitrariness of φ will show that we obtain a critical point of the renormalized energy, since the left-hand side is equal to 0. We prove (8.46) in Proposition 8.14 below.

We now provide the proof in the case in which the domain boundary is not flat, and so after flattening the boundary via a conformal transformation we obtain a family of functions in half space satisfying for all $R > 0$ the equation

$$\int_{B_R^+} \nabla w_{\varepsilon'} \cdot \nabla\varphi + \int_{\Gamma_R} a_{\varepsilon'}(x) \left(\frac{1}{2\varepsilon'} \sin 2w_{\varepsilon'} + h_{\varepsilon'}\right) \varphi = 0, \tag{8.47}$$

where ε' is defined just before (8.35). We follow the proof we presented above for the flat case, with some modifications. We define the stress energy tensor in the same way as in (8.44). We can then compute in the same way and we get:

$$
\begin{aligned}
\int_{\overline{\mathbb{R}^2_+}} \nabla\nabla\varphi : \hat{S}_{\varepsilon'} &= \int_{\mathbb{R}^2_+} \nabla\nabla\varphi : S_{\varepsilon'} - \frac{1}{\varepsilon'}\int_{\mathbb{R}} \varphi_{xx}\Psi\left(w_{\varepsilon'}\right) \\
&= -\int_{\mathbb{R}^2_+} \nabla\varphi\cdot \operatorname{div} S_{\varepsilon'} + \int_{\mathbb{R}} \nu\cdot S_{\varepsilon'}\nabla\varphi + \frac{1}{\varepsilon'}\int_{\mathbb{R}} \varphi_x\Psi'\left(w_{\varepsilon'}\right)\partial_x w_{\varepsilon'} \\
&= \int_{\mathbb{R}} \frac{\partial w_{\varepsilon'}}{\partial\nu}\nabla w_{\varepsilon'}\cdot\nabla\varphi - \frac{1}{2}\frac{\partial\varphi}{\partial\nu}|\nabla w_{\varepsilon'}|^2 + \frac{1}{\varepsilon'}\int_{R} \varphi_x\Psi'\left(w_{\varepsilon'}\right)\partial_x w_{\varepsilon'} \\
&= \int_{\mathbb{R}} \left(\frac{\partial w_{\varepsilon'}}{\partial\nu} + \frac{1}{\varepsilon'}\Psi'\left(w_{\varepsilon'}\right)\right)\partial_x w_{\varepsilon'}\varphi_x \\
&= \int_{\mathbb{R}} \left(\frac{\partial w_{\varepsilon'}}{\partial\nu} + \frac{a_{\varepsilon'}}{\varepsilon'}\Psi'\left(w_{\varepsilon'}\right) + a_{\varepsilon'}h_{\varepsilon'}\right)\partial_x w_{\varepsilon'}\varphi_x \\
&+ \int_{\mathbb{R}} \frac{1-a_{\varepsilon'}}{\varepsilon'}\Psi'\left(w_{\varepsilon'}\right)\partial_x w_{\varepsilon'}\varphi_x \\
&- \int_{\mathbb{R}} a_{\varepsilon'}h_{\varepsilon'}\partial_x w_{\varepsilon'}\varphi_x
\end{aligned} \tag{8.48}
$$

The first term vanishes because of the boundary condition satisfied by $w_{\varepsilon'}$. Using integration by parts we can then rewrite the right-hand side as:

$$
\begin{aligned}
-\int_{\mathbb{R}} \frac{1-a_{\varepsilon'}}{\varepsilon'}\Psi\left(w_{\varepsilon'}\right)\varphi_{xx} + \int_{\mathbb{R}} \frac{a'_{\varepsilon'}}{\varepsilon'}\Psi\left(w_{\varepsilon'}\right)\varphi_x \\
+ \int_{\mathbb{R}} \left(a_{\varepsilon'}h_{\varepsilon'}\right)_x w_{\varepsilon'}\varphi_x + \int_{\mathbb{R}} a_{\varepsilon'}h_{\varepsilon'}w_{\varepsilon'}\varphi_{xx}.
\end{aligned} \tag{8.49}
$$

We can show that the terms involving φ_{xx} converge to 0 using the H^2 weak convergence of $w_{\varepsilon'}$ away from the vortices and the convergence of the penalty term to 0 on any compact set bounded away from the vortices – as proved in (8.40) – such as $\operatorname{supp}\left(\varphi_{xx}\right)$, along with the locally uniform convergence $a_{\varepsilon'} \to 1$ and $h_{\varepsilon'} \to 0$. We also use the boundedness of $w_{\varepsilon'}$ and φ_x and locally uniform convergence $a_{\varepsilon'} \to 1$ and $h_{\varepsilon'} \to 0$ to show that the third term also converges to 0. To show convergence of the second term we use that $a'_{\varepsilon'} \to 0$, and the boundedness of the penalty term on $\operatorname{supp}\left(\varphi_x\right)$: indeed from the boundedness of the penalty term for the functions u_ε, i.e. $\frac{1}{2\varepsilon}\int_{\partial\Omega}\sin^2\left(u_\varepsilon - g\right)d\mathcal{H}^1 \leq C$ for some

$C > 0$, the definition of $w_{\varepsilon'}$ and rescaling we obtain that $\frac{1}{\varepsilon'}\int_{\operatorname{supp}(\varphi_x)}\Psi\left(w_{\varepsilon'}\right)d\mathcal{H}^1 \leq CK$ for some constant $K > 0$. Hence we have showed that $\int_{\overline{\mathbb{R}^2_+}}\nabla\nabla\varphi : \hat{S}_{\varepsilon'} \to 0$. The rest of the proof now is identical as above.

$\square$

Proposition 8.14. *We have:*

$$\int_{\mathbb{R}^2_+}\nabla\nabla\varphi : S_* = \pi\nabla W_{b_i}\left(b\right)\cdot\nabla\varphi. \tag{8.50}$$

Proof. For a radius $r > 0$, we write $\mathbb{R}^2_{+,r} := \mathbb{R}^2_+ \setminus B_r\left(b_i\right)$ and $\mathbb{R}_r := \mathbb{R} \setminus B_r\left(b_i\right)$. We now have that, for $r > 0$ small enough so that $\nabla\varphi$ is constant in the ball of radius r around b_i:

$$\begin{aligned}\int_{\mathbb{R}^2_+}\nabla\nabla\varphi : S_* &= -\underbrace{\int_{\mathbb{R}^2_{+,r}}\nabla\varphi\cdot\operatorname{div}S_*}_{=0} + \int_{\partial\mathbb{R}^2_{+,r}}\left(\nu\cdot S_*\right)\cdot\nabla\varphi \\ &= \int_{\mathbb{R}_r}\left(\nu\cdot S_*\right)\cdot\nabla\varphi + \int_{\underbrace{\partial B_r\left(b_i\right)\cap\mathbb{R}^2_+}_{:=\beta_i^r}}\left(\nu\cdot S_*\right)\cdot\nabla\varphi \\ &= \int_{\mathbb{R}_r}\frac{\partial u_*}{\partial\nu}\nabla u_*\cdot\nabla\varphi + \int_{\beta_i^r}\left(\nu\cdot S_*\right)\cdot\nabla\varphi \\ &= \int_{\beta_i^r}\left(\nu\cdot S_*\right)\cdot\nabla\varphi = c\int_{\beta_i^r}\left(\nu\cdot S_*\right)\cdot e_1,\end{aligned} \tag{8.51}$$

where in the last equality we have used that $\nabla\varphi = ce_1$ near b_i: this derives from the assumption that $\nabla\varphi$ is constant near b_i and that $\frac{\partial\varphi}{\partial\nu} = 0$ on the boundary, so the component of $\nabla\varphi$ in the e_2 direction is 0 near b_i. Observe in particular that this integral does not depend on r for r small enough, so we can make r as small as we want. Assume now without loss of generality (and to make our computation a bit easier to follow) that $b_i = 0$. We write $u_* = \arg\left(z\right) + R\left(z\right)$, where R is a smooth function in a neighbourhood of 0. Let $A = \nabla R\left(0\right)$. We then have, at distance r from 0, that $\nabla u_* = \nabla\arg\left(z\right) + A + O\left(r\right)$. The integral we need to compute can now be written as follows (we write β^r for β_i^r) - we leave out the terms involving $O(r)$ (since they collectively give a term which tends to 0 with r) to make our calculation easier to read:

$$\int_{\beta^r} (\nu \cdot (\nabla \arg(z) + A)) ((\nabla \arg(z) + A) \cdot e_1) - \frac{1}{2} (\nu \cdot e_1) |\nabla \arg(z)|^2. \quad (8.52)$$

The terms involving only A give an $O(r)$ term after integration, and the terms with $|\nabla \arg(z)|^2$ vanish due to symmetry. So we are only left to compute the terms involving cross products. We have:

$$\int_{\beta^r} (\nu \cdot A) (\nabla \arg(z) \cdot e_1) - (\nu \cdot e_1) (A \cdot \nabla \arg(z)). \quad (8.53)$$

Using polar coordinates, and observing that $\nu = r(\cos\theta, \sin\theta)$ and $\nabla \arg = \frac{1}{r}(-\sin\theta, \cos\theta)$ we get that this is equal to

$$\begin{aligned} &\int_0^\pi (A_1 \cos\theta + A_2)(-\sin\theta) - \cos\theta(-A_1 \sin\theta + A_2 \cos\theta)\, d\theta \\ &= \int_0^\pi \left(-A_1 \cos\theta \sin\theta - A_2 \sin^2\theta + A_1 \cos\theta \sin\theta - A_2 \cos^2\theta\right) d\theta \\ &= -\pi A_2 = \pi \nabla_{b_i} W(b). \end{aligned} \quad (8.54)$$

Now by sending r to 0, and remembering that $\nabla\varphi = ce_1$, we get that, using (8.51):

$$\int_{\mathbb{R}^2_+} \nabla\nabla\varphi : S_* = \nabla W_{b_i}(b) \cdot \nabla\varphi = \pi \nabla_{b_i} W(b)\, c. \quad (8.55)$$

□

We can now prove the main result of this chapter:

Theorem 8.15. *Let u_ε be a sequence of critical points for E_ε satisfying the energy bound $E_\varepsilon(u_\varepsilon) \leq M|\log\varepsilon|$ and such that the penalty term is bounded, i.e it satisfies $\frac{1}{2\varepsilon}\int_{\partial\Omega} \sin^2(u_\varepsilon - g)\, d\mathcal{H}^1 \leq C$ for some $C > 0$. Then $u_\varepsilon \rightharpoonup u_*$ in*

$W^{1,p}(\Omega)$, where u_* is an harmonic function such that $\sin(u_* - g) = 0$ on $\partial\Omega$ that has boundary jumps at points $a_i, i = 1, \dots, n$, where it jumps by $\pm\pi$.

Proof. All the results are known from [35, Theorem 8.6], except for the fact that we can only have single jumps. Recall from Lemma 8.3 that we can cover the set S_ε will balls $B_{\sigma\varepsilon}(a_k^\varepsilon), a_k^\varepsilon \in \partial\Omega$ such that $|a_i^\varepsilon - a_j^\varepsilon| \gg \varepsilon$, the degree of the transition around a_k^ε is ± 1 and such that for every k we have $a_k^\varepsilon \to a_k$. Assume that the jump of the limit function u_* for one of these limit point is $\pm d\pi$ with $d > 1$ for some point $a_{i_0}, i_0 \in \{1, \dots, n\}$. Then there would be at least two sequences $a_k^\varepsilon, k = 1, \dots, N$ which converge to a_{i_0}. Let $I \subset \{1, \dots, N\}$ be the set of all and only the indices for which $a_i^\varepsilon \to a_{i_0}$. Our goal is to show that the set I consists of a single element. By our construction we have that:

$$|a_i^\varepsilon - a_j^\varepsilon| \gg \varepsilon \quad \text{for } i, j \in I, i \neq j.$$

As in Lemma 8.7 we can partition the points $a_i^\varepsilon, i \in I$ (and correspondingly the points α_i^ε) according to their asymptotic distance. We then obtain that the distances are on levels $\varepsilon \ll \lambda_1^\varepsilon \ll \dots \ll \lambda_k^\varepsilon$ for some $1 \leq k \leq |I|$. Let $\{J_s^t\}_s$ be a partition of I such that for $i.j \in J_s^t$ the distance of a_i^ε from a_j^ε is at most of order λ_t^ε: we can obtain this considering the equivalence relation according to which $i \sim j$ if and only if there is a constant such that $|a_j^\varepsilon - a_i^\varepsilon| \leq C\lambda_t^\varepsilon$ for all ε small enough. Reflexivity and symmetry are obvious and transitivity follows from the triangle inequality. In particular we observe that for i, j in different elements of the partition on scale λ_t we have that $|a_i^\varepsilon - a_j^\varepsilon| \gg \lambda_t^\varepsilon$. We want to show that for each such partition the cardinality of each partition element is 1, which will prove our conclusion (observe that $\{J_s^k\}_s = \{I\}$, since all points have at most a distance of order λ_k). We do this by finite induction. Start at level λ_1. Consider one element J_s^1 of the partition. We flatten the boundary via a conformal transformation – which transforms the points a_i^ε into the points α_i^ε (recall what we said after (8.27)). We then perform a blow-up around one point $\alpha_{k_0}^\varepsilon$ for some $k_0 \in J_s^1$ (it is not important which for our proof) at scale

λ_1 defining a sequence of functions $w_{\varepsilon'}$ as in (8.35). By Proposition 8.10 these converge weakly in $W^{1,p}_{loc}\left(\mathbb{R}^2_+\right)$ to a harmonic function defined in half-space (as we saw in Proposition 8.10) which has boundary jumps $d_k\pi$ at points β_k for $k = 1, \ldots, K$. We now show that $d_k \in \{+1, -1\}$: thanks to Lemma 8.3 around each point a_i^ε we have a transition of degree ± 1 and the points a_i^ε are at a distance $\gg \varepsilon$. Both of these remain true after flattening the boundary for the transition around the points α_i^ε. Since now we are looking at only those indices which lie in one partition element J_s^1 we have that there exist constants $C_1, C_2 > 0$ such that $C_1\lambda_1^\varepsilon \leq |\alpha_i^\varepsilon - \alpha_j^\varepsilon| \leq C_2\lambda_1^\varepsilon$: the second inequality derives from the definition of our partition, the first from the way that we defined λ_1^ε (recall Lemma 8.7) and the fact that it is the smallest length scale among the distances between the points. After blow-up on the scale λ_1^ε this means that for the points $\beta_k^\varepsilon := \left(\alpha_k^\varepsilon - \alpha_{k_0}^\varepsilon\right)/\lambda_1^\varepsilon$ we have $C_1 \leq |\beta_i^\varepsilon - \beta_j^\varepsilon| \leq C_2$, and around each of these points the degree of the transition is ± 1 (since it was for the α_k). Thus we can conclude that the points β_k^ε converge to distinct limits β_k (since their distance is bounded from below by C_1) and that the degree d_k is ± 1. By Theorem 8.13 the point $(\beta_1, \ldots, \beta_K)$ is a critical point for the half-space renormalized energy $W\left(b_k, d_k\right) = -\pi \sum_{i \neq j} d_j d_i \log|b_i - b_j|$, where $d_k \in \{-1, +1\}$ thanks to what we just said. Let P and Q be complex polynomials such that the roots of P are the points with degree $+1$ and the roots of Q are the points with degree -1, and such that no root is repeated. By Theorem 8.16 they satisfy Tkachenko's equation. Then we can apply the results in Section 8.4 and in particular by Theorem 8.21 we get that since in our case all points lie on a straight line, we cannot have more than one point. So there is only one point in J_s^1 for every s, i.e. every partition element contains a single point. In particular we have proved that the centres a_i^ε of the balls $B_{\sigma\varepsilon}\left(a_i^\varepsilon\right)$ in our cover of S_ε have an asymptotic distance at least of order λ_2, and around each ball $B_{\sigma\varepsilon}\left(a_i^\varepsilon\right)$ we have a degree of ± 1 (since we have a single element in each J_s^1). So now we are in the same situation as at the start, only now the minimal distance between the a_i^ε is of order λ_2. Since now λ_2 is the smallest length scale we see that all the result in this chapter apply to blow-ups on scale λ_2. We can then repeat the argument we just carried out to show that each partition element J_s^2 only contains one point. By repeating this procedure at all scales λ_k – where at each step the number of such length scales is reduced by 1 – we get that for every k each partition element J_s^k contains a single element. From this, and recalling that $\{J_s^k\}_s = \{I\}$ (i.e. the partition at scale k only consists of the set I), we obtain that $|I| = 1$, which concludes the proof. □

8.4 Appendix: Equilibrium of charges in the complex plane

In this section we collect some results which we need to prove the single multiplicity result of this chapter, which are due to Tim Espin and should be published as part of a joint paper with me and Matthias Kurzke (see [5]). The main idea is that we can describe the position of our points as roots of a special kind of polynomials, which are called *Adler-Moser polynomials*. Then with the help of these we can show that if the points lie on a straight line, then there can only be one, which is what we need to prove the single multiplicity result of this chapter.

If P and Q are complex polynomials we say that they satisfy Tkachenko's equation if

$$P''Q + Q''P = 2P'Q'. \tag{8.56}$$

We consider some special solutions to this equation, namely the Adler-Moser (A-M) polynomials: these are a sequence of polynomial solutions to Tkachenko's equation with non-repeated roots. They are a convenient way of studying the equilibrium of charges problem in the whole complex plane. In this section we recall and prove some results about the positions of their roots in the complex plane, and also make observations regarding the symmetries of their roots. Our main reference for this section is [3].

Given a system of N charges at positions $a_1, ..., a_N \in \mathbb{C}$ with charges $d_1, ..., d_N$, the system of N conditions for a critical point in the energy of the system, and hence an equilibrium is

$$\sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{d_j}{a_k - a_j} = 0 \tag{8.57}$$

for each $k = 1, ..., N$. Throughout, we assume that each $d_j \in \{1, -1\}$. In the case of n_+ positive charges at positions $z_1, ..., z_{n_+}$ and n_- negative charges at positions $\zeta_1, ..., \zeta_{n_-}$, the stationarity conditions are

$$\sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n_+} \frac{1}{z_\alpha - z_\beta} = \sum_{\lambda=1}^{n_-} \frac{1}{z_\alpha - \zeta_\lambda} \tag{8.58}$$

for all $\alpha = 1, ..., n_+$, and

$$\sum_{\alpha=1}^{n_+} \frac{1}{\zeta_\lambda - z_\alpha} = \sum_{\substack{\mu=1 \\ \mu \neq \lambda}}^{n_-} \frac{1}{\zeta_\lambda - \zeta_\mu} \tag{8.59}$$

for all $\lambda = 1, ..., n_-$.

The following is a widely known result, found for instance in [3] and [15].

Theorem 8.16. *Let P and Q be complex polynomials of degrees n_+ and n_- respectively, with no repeated roots. Suppose the roots of P represent the positions of n_+ positive $+1$ charges and the roots of Q represent the positions of n_- negative -1 charges. Then the following statements are equivalent:*

(i) The system of point charges represented by the $N = n_+ + n_-$ roots of P and Q is in equilibrium;

(ii) The polynomials P and Q satisfy Tkachenko's equation,

$$P''Q + Q''P = 2P'Q'\,.$$

Proof. To prove sufficiency, one can follow the derivation in Section 3 of [3]. The proof of necessity was given by Tim Espin. For the details we refer to [5]. In this thesis we only need sufficiency (i.e. that (i) implies (ii).) $\square$

Remark 7. Note that when we balance the highest order terms of P and Q in Tkachenko's equation, we obtain the well-known result that the degrees n_+ and n_-, and therefore the number of positive and negative charges, must be successive triangular numbers, see for example [15] at the start of Section 4.

In what follows, we assume the Adler-Moser polynomials to be normalized so that their leading coefficient is 1. We have the following important classification result:

Proposition 8.17. *The Adler-Moser polynomials are the unique (up to constant of integration) polynomial solutions of Tkachenko's equation when there are no repeated roots.*

Proof. This is proved by Burchnall and Chaundy in [11]. $\square$

Remark 8. In their paper, Burchnall and Chaudy conjecture that the Adler-Moser polynomials are the only polynomial solutions to Tkachenko's equation, but this is shown to be false by Demina and Kudryashov in [15] when polynomials with repeated roots are allowed. As stated before, however, we are only interested in the case with no repeated roots.

It is a known fact that Adler-Moser polynomials can be generated recursively, see for instance the paper by Adler and Moser [2] where they introduce the polynomials or that by Loutsenko [41]:

Proposition 8.18. *The Adler-Moser polynomials P_n can be generated recursively as follows. Let $P_0(z) := 1$ and $P_1(z) := z$. Then the polynomial P_n has degree $\deg P_n = \frac{n(n+1)}{2}$ (so the degrees of the n-th polynomial is the n-th* *triangular number) and can be found solving the following differential equation:*

$$P'_{n+1}P_{n-1} - P'_{n-1}P_{n+1} = (2n+1)P_n^2. \tag{8.60}$$

At each step a constant of integration is introduced.

We can list some of the first polynomials (this will also be important for the proof of Proposition 8.19 below): we have (see [41], where t_i denotes the constant of integration we choose at each step):

$$\begin{aligned} &P_0(z) = 1, \\ &P_1(z) = z, \\ &P_2(z) = z^3 + t_2, \\ &P_3(z) = z^6 + 5t_2z^3 + t_3z - 5t_2^2 \\ \dots \end{aligned} \tag{8.61}$$

The following results (Proposition 8.19, Corollary 8.20 and Theorem 8.21) were proved by Tim Espin and will appear in a joint paper with myself and Kurzke. We report the results and the proofs here:

Proposition 8.19. *The coefficients of the second and third highest order terms in the Adler-Moser polynomials of degree higher than one are both zero.*

Proof. We proceed using proof by strong induction on $n \geq 2$. Let $S(n)$ be the statement that the coefficient of the second and third highest order terms in the n-th A-M polynomial is zero. The $n=2$ and $n=3$ cases clearly hold by (8.61). Suppose $S(k-1)$ and $S(k)$ hold for some $k > 2$, and consider $S(k+1)$. We compare coefficients in the recurrence relation (8.60) for generating A-M polynomials. By the inductive hypothesis,

$$P_{k-1} = z^{(k^2-k)/2} + 0z^{(k^2-k-2)/2} + 0z^{(k^2-k-4)/2} + \dots,$$

$$P'_{k-1} = \frac{k^2-k}{2}z^{(k^2-k-2)/2} + 0z^{(k^2-k-4)/2} + 0z^{(k^2-k-6)/2} + \dots,$$

$$
\begin{aligned}
P_k &= z^{(k^2+k)/2} + 0z^{(k^2+k-2)/2} + 0z^{(k^2+k-4)/2} + \dots\,, \\
(2k+1)P^2 &= (2k+1)z^{k^2+k} + 0z^{k^2+k-2} + 0z^{k^2+k-2} + \dots\,,
\end{aligned}
$$

and

$$
\begin{aligned}
P_{k+1} &= z^{(k^2+3k+2)/2} + a_1 z^{(k^2+3k)/2} + a_2 z^{(k^2+3k-2)/2} + \dots\,, \\
P'_{k+1} &= \frac{k^2+3k+2}{2} z^{(k^2+3k)/2} + a_1 \frac{k^2+3k}{2} z^{(k^2+3k-2)/2} \\
&\qquad + a_2 \frac{k^2+3k-2}{2} z^{(k^2+3k-4)/2} + \dots\,,
\end{aligned}
$$

where a_1 and a_2 are undetermined coefficients which we want to show to be zero. Now,

$$
\begin{aligned}
P'_{k+1}P_{k-1} &= \frac{k^2+3k+2}{2} z^{k^2+k} + a_1 \frac{k^2+3k}{2} z^{k^2+k-1} + a_2 \frac{k^2+3k-2}{2} z^{k^2+k-2} + \dots\,, \\
P'_{k-1}P_{k+1} &= \frac{k^2-k}{2} z^{k^2+k} + a_1 \frac{k^2-k}{2} z^{k^2+k-1} + a_2 \frac{k^2-k}{2} z^{k^2+k-2} + \dots\,.
\end{aligned}
$$

Thus by comparing the three highest order terms on each side of $P'_{k+1}P_{k-1} - P'_{k-1}P_{k+1} = (2k+1)P_k^2$, we have:

$$
\frac{k^2+3k+2}{2} - \frac{k^2-k}{2} = 2k+1\,,
$$

which is clearly satisfied;

$$
a_1\frac{k^2+3k}{2} - a_1\frac{k^2-k}{2} = 0 \implies a_1 = 0\,;
$$

and

$$
a_2\frac{k^2+3k-2}{2} - a_2\frac{k^2-k}{2} = 0 \implies a_2 = 0\,.
$$

Thus the coefficients of the second and third highest powers of z in P_{k+1} are zero, so the statement $S(k+1)$ holds. Therefore $S(n)$ holds by induction for all $n \geq 2$ as required. □

Corollary 8.20. *Given a particular Adler-Moser polynomial of degree higher than one, there is no straight line in the complex plane which passes through all its roots.*

Proof. Let $n \geq 2$. Suppose the distinct roots of $P_n(z)$ are $z_1, ..., z_n \in \mathbb{C}$ (since P_n is an Adler-Moser polynomial it has no repeated roots by definition). Assume for a contradiction that these all lie on a straight line in the complex plane. $P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + ...$, where $a_1 = -(z_1 + ... + z_n)$ and $a_2 = z_1 z_2 + z_1 z_3 + ... + z_{n-1} z_n$. By Proposition 8.19, $a_1 = a_2 = 0$, which means that $z_1^2 + z_2^2 + ... + z_n^2 = a_1^2 - 2a_2 = 0$. This implies that not all the roots are real, as they cannot be all zero. Since $a_1 = 0$, the straight line passing through all the roots must also pass through zero. Therefore there exists an unique non-zero $\varphi \in (-\pi/2, \pi/2]$ such that the roots lie on the line $\Im(z) = \Re(z) \tan \varphi$ in the complex plane if $\varphi \neq \pi/2$, and if $\varphi = \pi/2$ the line is the imaginary axis. For $j = 1, ..., n$ define the points $y_j = e^{-i\varphi} z_j \in \mathbb{R}$, and the new complex polynomial $R(z)$ by

$$R(z) = (z - y_1)(z - y_2)...(z - y_n) = z^n + b_1 z^{n-1} + b_2 z^{n-2} + ...$$

Here, $b_1 = e^{-i\varphi} a_1 = 0$ and $b_2 = e^{-2i\varphi} a_2 = 0$. This implies $y_1^2 + y_2^2 + ... + y_n^2 = 0$, and since the y_j are all real, we must have $y_1 = ... = y_n = 0$. Thus, since $\varphi \neq 0$, $z_j = 0$ for all j, which is a contradiction. Therefore the roots of P_n for $n \geq 2$ cannot all lie on a straight line. $\square$

A consequence of this is the next result, which we used in the proof of Theorem 8.15:

Theorem 8.21. *If we have a configuration of ± 1 charges in equilibrium which all lie on a straight line, then the total number of charges is one.*

Proof. Consider a stationary configuration of ± 1 charges in the complex plane. By Proposition 8.17 and Remark 7, the positions of the two species are described by the roots of two successive A-M polynomials. If all the charges lie on the same straight line, then the roots of each of these polynomials individually must also lie on the same straight line. By Corollary 8.20, the highest degree polynomial must have degree no higher than 1. Therefore the two A-M polynomials here are P_0 and P_1, so there must be only one charge (which may be either positive or negative) in the plane. $\square$

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